İstanbul Üniv. Fen Fak. Mec. Seri A, 43 (1978), 151-157

ON SPECIAL CURVES OF A FINSLER SPACE

C. M. PRASAD - R. K. SRIVASTAVA

In this paper it has been established results relating to special curves in a hyper-surface of a Finsier space. It has also been obtained the conditions when a special curve reduces to union, hyperasymptotic and hyper-normal curves and has been given many properties in general and in special case as well.

1. INTRODUCTION

Consider a hyper-surface F_{n-1} given by the equation $x^i = x^i(u^\alpha)$, $i = 1,2,\ldots,n$; $\alpha = 1,2,\ldots,n-1$, u being the Gaussian coordinates, be immersed in an n-dimensional Finsler space F_n. Consider a vector $x'^i = \dfrac{dx^i}{ds}$ (of magnitudes $F(x, x')$) tangent to F_n such that $x^i = B_\alpha^i u'^\alpha$, where $B_\alpha^i = \dfrac{\partial x^i}{\partial u^\alpha}$. The fundamental metric tensors $g_{ij}(x, x')$ and $g_{\alpha\beta}(u, u')$ of F_n and F_{n-1} are related by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') B_\alpha^i B_\beta^j . \tag{1.1}$$

There exists a unit vector n^{*i} called the secondary normal, normal to F_{n-1} [3] and satisfies the following equations:

$$g_{ij}(x, x') n^{*i} B^j = n_j^* B^j = 0 \tag{1.2}$$

$$g_{ij}(x, n^*) n^{*i} n^{*j} = 1 \tag{1.3}$$

$$g_{ij}(x, x') n^{*i} n^{*j} = \Psi . \tag{1.4}$$

Let $C: u^\alpha = u^\alpha(s)$ be a curve in F_{n-1}, s being the arc length of the curve. If q^i and p^α are the components of the first curvature of the curve with respect to F_n and F_{n-1} respectively, we have

$$q^i = p^\alpha B_\alpha^i + K_{n^*} n^{*i} , \tag{1.5}$$

where

$$K_{n^*} = \Omega^*_{\alpha\beta}(u, u') \left(\frac{du^\alpha}{ds}\right) \left(\frac{du^\beta}{ds}\right) \tag{1.6}$$

is called the secondary normal curvature of the hyper-surface and $\Omega^*_{\alpha\beta}(u, u')$ are the components of the secondary second fundamental tensor of the hyper-surface.

The geodesic curvatures of F_{n-1} and F_n and the secondary normal curvature of F_{n-1} are related by

$$K_a^2 = K_g^2 + K^2_{n^*} \,, \tag{1.7}$$

where $K_a^2 = g_{ij}(x, x') \, q^i q^j$, $K_g^2 = g_{\alpha\beta} \, p^\alpha p^\beta$ are the geodesic curvatures of F_n and F_{n-1} respectively.

Consider a congruence λ^i of curves such that one curve of it passes through each point of the hyper-surface. We may write

$$\lambda^i = t^\alpha B_\alpha^i + \Gamma \, n^{*i} \,, \tag{1.8}$$

where we have

$$\Gamma \Psi = g_{ij}(x, x') \, n^{*j} \lambda^i \tag{1.9}$$

and

$$t_\beta = g_{\alpha\beta}(u, u') \, t^\alpha = g_{ij}(x, x') \, \lambda^i B_\beta^j \,. \tag{1.10}$$

The vector λ^i is normalized by the equation $g_{ij}(x, x') \, \lambda^i \lambda^j = 1$.

If θ is the angle between the vectors λ^i and n^{*i}, then we have

$$\Gamma \sqrt{\Psi} = \cos\theta = \frac{g_{ij}(x, x') \, \lambda^i n^{*j}}{\sqrt{\Psi}} \quad \text{and} \quad t^\alpha t_\alpha = \sin^2\theta \,. \tag{1.11}$$

If C is a special curve in F_{n-1}, we have [1]

$$\lambda^i = a \, p^i + b \, q^i \,, \tag{1.12}$$

where a and b are parameters and $p^i = p^\alpha B_\alpha^i$.

2. SPECIAL CURVES IN A HYPER-SURFACE OF A FINSLER SPACE

In view of the equations (1.5), (1.8) and (1.12), we have

$$t^\alpha B_\alpha^i + \Gamma \, n^{*i} = a \, p^\alpha B_\alpha^i + b \, (p^\alpha B_\alpha^i + K_{n^*} n^{*i}) \,. \tag{2.1}$$

Multiplying (2.1) by $g_{ij} B_\beta^j p^\beta$, we have

$$g_{\alpha\beta} \, t^\alpha p^\beta = (a + b) \, K_g^2 \,. \tag{2.2}$$

But

$$g_{\alpha\beta} \, t^\alpha p^\beta = K_g \sin\theta \cos\beta \,, \tag{2.3}$$

where β is the angle between the vector with contravariant components t^α and the first curvature vector of the special curve with respect to F_{n-1} and their magnitudes are $\sin\theta$ and K_g respectively.

Substituting from equations (1.11) and (2.3) for $\Gamma\sqrt{\Psi}$ and $g_{\alpha\beta}\, t^{\alpha} p^{\beta}$ respectively in the differential equation of special curve

$$p^{\alpha} - t^{\alpha}(1 - \Gamma^2 \Psi)^{-1} g_{\beta\gamma}\, t^{\beta} p^{\gamma} = 0,$$

which was given by Prasad [1], we have

$$p^{\alpha} - t^{\alpha} K_g \cos\beta \operatorname{cosec}\theta = 0 \tag{2.4}$$

as the alternative form of the differential equation of the special curve in a hyper-surface of Finsler space.

From (2.4), we have :

Theorem 2.1. Each geodesic on the hyper-surface F_{n-1} of F_n is a special curve.

Equating (2.2) and (2.3), we get

$$(a + b) K_g = \sin\theta \cos\beta . \tag{2.5}$$

From (2.5), we have :

Theorem 2.2. If λ^i is not normal to F_{n-1}, the condition for a special curve to be a geodesic in F_{n-1} is that, t^{α} is orthogonal to the first curvature vector of the special curve with respect to F_{n-1}.

Remark. In view of the theorem 2.1, the condition of the theorem 2.2 is sufficient also.

Multiplying (1.8) by $g_{ij}(x, x') n^{*j}$, we get

$$\Gamma\sqrt{\Psi} = \sqrt{\Psi}\, b\, K_{n^*} = \cos\theta , \tag{2.6}$$

since $\Gamma\sqrt{\Psi} = \cos\theta$.

From (2.6) and (1.7), we have

$$(a + b)^2 K_a{}^2 = \Psi K^2{}_{n^*} [(a + b)^2 - b^2 \cos^2\beta] + \cos^2\beta . \tag{2.7}$$

Now we consider the following particular cases :

If $\beta = \pi/2$, then

$$K_a = \sqrt{\Psi}\, K_{n^*} , \tag{2.8}$$

from which we have the following :

Theorem 2.3. When t^{α} is orthogonal to the first curvature vector with respect to F_{n-1}, the geodesie curvature of the special curve with respect to F_n is proportional to its normal curvature.

Theorem 2.4. When t^α is orthogonal to p^α, the necessary and sufficient condition for a special curve to be asymptotic line in F_{n-1}, is that it is a geodesic in F_n.

Proof. Since $\beta = \pi/2$ and let special curve is asymptotic line in F_{n-1}, then $K_{n^*} = 0$ and therefore from (2.8) $K_a = 0$.

Conversely, if $\beta = \pi/2$ and special curve is a geodesic in F_n, $K_a = 0$, and since $\Psi \neq 0$, $K_{n^*} = 0$ from (2.8), which completes the proof.

Squaring equation (1.12), we have

$$1 = a^2 g_{\alpha\beta} p^\alpha p^\beta + b^2 (K_g{}^2 + \Psi K^2{}_{n^*}) + 2\, ab\, K_g{}^2$$

or

$$1 = (a+b)^2 K_g{}^2 + b^2 \Psi K^2{}_{n^*}. \tag{2.9}$$

From the above equation we have the following:

Theorem 2.5. For an asymptotic line, the geodesic curvatures of the special curve is constant.

Proof. Since, for an asymptotic line $K_{n^*} = 0$, therefore the proof follows from the equation (2.9).

Theorem 2.6. If the special curve is a geodesic, then the secondary normal curvature will be constant.

Proof. For a geodesic, $K_g = 0$, therefore the proof follows from the equation (2.9).

3. SPECIAL CURVE AND THE UNION CURVE

Let λ^i be the contravariant components in F_n of a unit vector field in the direction of a congruence of curves, then for a union curve of F_{n-1} we have [4]

$$\lambda^i = w\, B_\alpha{}^i \frac{du^\alpha}{ds} + uq^i, \tag{3.1}$$

where w and u are parameters. Equating (1.8) and (3.1) and multiplying the resulting equation by $g_{ij} B_\beta{}^j$ and $g_{ij} n^{*i}$ respectively, we get

$$\text{a)} \quad t^\alpha = w \frac{du}{ds} + u p^\alpha, \quad \text{b)} \quad \Gamma = u K_{n^*}. \tag{3.2}$$

For a spesial curve, we have obtained from equation (1.8) and (1.12)

$$\text{a)} \quad t^\alpha = (a+b) p^\alpha, \qquad \text{b)} \quad \Gamma = b K_{n^*}. \tag{3.3}$$

Comparing (3.2b) and and (3.3b), we get

$$u = b \,. \tag{3.4}$$

Again comparing (3.2a) and (3.3a) and using (3.4), we get

$$w \frac{du^\alpha}{ds} = a\, p^\alpha \,. \tag{3.5}$$

The relation (3.5) implies that when $w = 0$, either $p^\alpha = 0$ or $a = 0$. Thus, when $p^\alpha = 0$, we have:

Theorem 3.1. A special curve is a union curve, if it is a geodesic in the hyper-surface F_{n-1} and in this case the congruence λ^i is along the first curvature vector of F_n.

When a = 0, equation (2.7) in view of (2.6), reduces to $b\, K_g = \cos\beta \sin\theta$. Hence in view of equations (3.1) and (1.12) we have:

Theorem 3.2. If λ^i is not normal to F_{n-1}, a necessary and sufficient condition for a special curve to be the union curve is that t^α is orthogonal to the first curvature vector of special curve with respect to F_{n-1}.

4. SPECIAL CURVE AND HYPERASYMPTOTIC CURVE

The equation

$$\Psi\, K_{n^*} + g_{\alpha\beta}\, t^\alpha\, p^\beta / \Gamma = 0 \tag{4.1}$$

represents the differential equation of the hyperasymptotic curve in the hypersurface F_{n-1} of a Finsler space as defined by Prasad [2].

Comparing (4.1) and (2.2) and using equation (1.11) we have

$$(a + b)\, K_g^2 = -\sqrt{\Psi} \cos\theta\, K_{n^*} \,. \tag{4.2}$$

From (4.2) we may establish the following:

Theorem 4.1. The ratio of the square of the geodesic curvature of the special curve with respect to F_{n-1} and the secondary normal curvature of F_{n-1} varies as the cosine of the angle between the unit vector field and the secondary normal of the hyper-surface F_{n-1}.

When θ = constant, we have:

Theorem 4.2. The square of the geodesic curvature of the special curve with respect to F_{n-1} is directly proportional to the normal curvature of the F_{n-1} when the angle between the unit vector field λ^i and the secondary normal of the hyper-surface F_{n-1} remains constant throughout.

Equation (4.2) in view of equations (1.11) and (2.6) yields

$$(a + b)\, K_g^{\,2} = -\sqrt{\Psi}\, b\, K^2_{n^*} .$$

From the above equation we have the following:

Theorem 4.3. The square of the geodesic curvature of the specia l curve with respect to F_{n-1} is directly proportional to the secondary norma l curvature of F_{n-1}.

Let us define the resolved part of the curvature vector q^i in the direction of the congruence as [2]

$$K_c^* = g_{ij}(x, x')\, \lambda^i q^j . \tag{4.3}$$

Multiplying equation (1.12) by $g_{ij}(x, x')\; q^j$ and making use of 4.3) we have

$$K_c^* = a\; K_g^{\,2} + b\, K_a^{\,2} . \tag{4.4}$$

When $K_c^* = 0$, the special curve will be a hyperasymptotic curve [2]. Hence we have the following:

Theorem 4.4. A spec'al curve in the hyper-surface F_{n-1} is a hyperasymptotic curve in F_n if the enveloping space is totally geodesic.

Remark. If a $= b = 1$, the condition of the theorem 4.4 is sufficient as well.

Theorem 4.5. If t^α is perpendicular to p^α, and special curve is the hyper-asymptotic curve, then each geodesic is an asymptotic line.

Proof. If t^α is perpendicular to p^α, then from equation (2.4) $p^\alpha = 0$ and under the condition of the theorem from equation (4.1) $K_{n^*} = 0$. Therefore $p^\alpha = K_{n^*} = 0$ which completes the proof.

5. SPECIAL CURVE AND HYPERNORMAL CURVE

For a hyper-surface, the differential equation of the hypernormal curve [5] takes the form

$$g_{ij}(x, x')\, \lambda^i \frac{dx^j}{ds} = 0 .$$

As a consequence of equation (1.8), this can be written as

$$g_{\alpha\beta}\, t^{\alpha} \frac{du^{\beta}}{ds} = 0 . \tag{5.1}$$

Multiplying (1.12) by $g_{ij} \dfrac{dx^j}{ds}$, we have

$$g_{ij}(x, x')\, \lambda^i \frac{dx^j}{ds} = 0 , \tag{5.2}$$

where we have used the fact that $g_{\alpha\beta}\, p^{\alpha} \dfrac{du^{\beta}}{ds} = 0$ and

$$g_{ij}\, q^i \frac{dx^j}{ds} = 0.$$

From equation (5.2) and the definition of a hypernormal curve, it is obvious that each special curve is hypernormal curve.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR U.P.
INDIA

ÖZET

Bu çalışmada, bir Finsler uzaymdaki bir hiperyüzeye ait özel eğrilere ilişkin bazı sonuçlar elde edilmektedir. Aynı zamanda, özel bir eğrinin, hiperasimptotik ve hipernormal eğrilerin birleşimine indirgenebilmesi koşulları saptanmakta ve genel ve özel hallerde birçok özellikler verilmektedir.