ON SPECIAL PROJECTIVE MOTION IN AN SPR Fn-SPACE

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In this paper it has been established that in view of a certain equation and definition, Berwald's curvature tensor satisfies the so-called definition of recurrency.

1. INTRODUCTION

Let Fn [7][1] be an n-dimensional Finsler space equipped with a symmetric projective connection coefficient $\pi^i_{jk}(x, \dot{x}) \stackrel{\text{def.}}{=} \left\{ G^i_{jk} - \frac{1}{(n+1)} (2\delta^i_{(j} G^\gamma_{k)\gamma} + \right.$ $\left. + \dot{x}^i G^\gamma_{\gamma kj}) \right\}$. The so-called projective covariant derivative [6] of my tensor field $T^i_j(x, \dot{x})$ with respect to a $\pi^i_{jk}(x, \dot{x})$ is given by

$$T^i_{j((k))} = \partial_k T^i_j - \dot{\partial}_h T^i_j \pi^h_{sk} \dot{x}^s + T^s_j \pi^i_{sk} - T^i_s \pi^s_{jk} . \tag{1.1}$$

The well known commutation formula involving the above covariant derivative is given by

$$2T^i_{j[((h))((k))]} = - \dot{\partial}_\gamma T^i_j Q^\gamma_{shk} \dot{x}^s T^s_j Q^i_{shk} - T^i_s Q^s_{jhk}{}^{2)} , \tag{1.2}$$

where

$$Q^i_{hjk}(x, \dot{x}) \stackrel{\text{def.}}{=} 2\{\partial_{[k} \pi^i_{j]h} - \pi^i_{\gamma h[j} \pi^\gamma_{k]} + \pi^\gamma_{h[j} \pi^i_{k]\gamma}\} \tag{1.3}$$

is called projective entity and satisfies the following relations [6]:

$$Q^i_{hjk} = - Q^i_{hkj} \tag{1.4}$$

and

$$Q^i_{hji} = Q_{hj} . \tag{1.5}$$

Misra [6] has also obtained two more important relations:

$$Q^i_{jkh}(x, \dot{x}) = H^i_{jkh} + \frac{1}{(n+1)} (\delta^i_j H^\gamma_{\gamma hk} + \dot{x}^i \dot{\partial}_j H^\gamma_{\gamma hk}) +$$
$$+ \frac{2}{(n+1)^2} \left\{ (n+1) G^\gamma_{\gamma j[(k)} \delta^i_{h]} + \delta^i_{[h} \dot{\partial}_{k]} (G^\gamma_{\gamma j} G^s_s) \right\} \tag{1.6}$$

and

[1] Numbers in square brackets refer to the references given at the end of the paper.

[2] $2A_{(hk)} = A_{hk} + A_{kh}$ and $2A_{[hk]} = A_{hk} - A_{kh}$.

$$W^i{}_{hjk} = Q^i{}_{hjk} - \frac{2}{(n^2-1)}\Big\{(n+1)\, Q_{j[k} - H_{j[k} - H_{[k\langle j\rangle} + (n-1)\,\dot\partial_j\,\dot\partial_{[k}\, H - \\ - \dot x^s\,\dot\partial_j\, H^\gamma{}_{\gamma s[k}\Big\}\,\delta^i{}_{h]}{}^{3)}, \tag{1.7}$$

where

$$H^i{}_{hjk}(x, \dot x) = 2\{\partial_{[k}\, G^i{}_{j]h} - G^i{}_{\gamma h[j}\, G^\gamma{}_{k]} + G^\gamma{}_{h[j}\, G^i{}_{k]\gamma}\} \tag{1.8}$$

and

$$W^i{}_{hjk}(x, \dot x) = H^i{}_{hjk} + \frac{1}{(n+1)}\Big\{\delta_h{}^i\, H^\gamma{}_{\gamma kj} + \dot x^i\,\dot\partial_h\, H^\gamma{}_{\gamma kj} + \\ + 2\delta^i{}_{[j}\,(H_{\langle h\rangle k]} + \dot\partial_{k]}\,\dot\partial_h\, H)\Big\} \tag{1.9}$$

are called Berwald's and projective curvature tensor respectively. They also satisfy the following relations [7]:

$$\text{a) } H^i{}_{hjk} = - H^i{}_{hkj} \text{ and } \text{b) } H^i{}_{hji} = H_{hj} \tag{1.10}$$

and

$$\text{a) } W^i{}_{hjk} = - W^i{}_{hkj}, \text{ b) } W^i{}_{hji} = 0 \text{ and } \text{c) } W^i{}_{ihj} = 0. \tag{1.11}$$

Let us consider an infinitesimal point transformation

$$\bar x^i = x^i + v^i(x)\, dt, \tag{1.12}$$

where $v^i(x)$ means any vector field and dt is an infinitesimal point constant. By virtue of projective covariant derivative and above point transformation the so-called Lie-derivatives [9] of $T_j^i(x, \dot x)$ and $\pi^i{}_{hk}(x, \dot x)$ are given by

$$\pounds v\, T_j^i = T^i{}_{j((h))}\, v^h + \dot\partial_h\, T_j^i\, v^h{}_{((s))}\,\dot x^s + T_s^i\, v^s{}_{((j))} - T_j^s\, v^i{}_{((s))} \tag{1.13}$$

and

$$\pounds v\, \pi^i{}_{jk} = v^i{}_{((j))((k))} + Q^i{}_{jkh}\, v^h + \pi^i{}_{sjk}\, v^s{}_{((\gamma))}\,\dot x^\gamma \tag{1.14}$$

The following well known commutation formulae involving the projective covariant derivative and Lie-derivatives are given by

$$\pounds v\,(T^i{}_{j((s))}) - (\pounds v\, T_j^i)_{((s))} = (\pounds v\, \pi^i{}_{sj})\, T_j^h - (\pounds v\, \pi^h{}_{sj})\, T_h^i - (\pounds v\, \pi^h{}_{sm})\,\dot x^m\,\dot\partial_h\, T_j^i \tag{1.15}$$

and

$$(\pounds v\, \pi^i{}_{jh})_{((k))} - (\pounds v\, \pi^i{}_{kh})_{((j))} = \pounds v\, Q^i{}_{hjk} + 2\dot x^s\, \pi^i{}_{\gamma h[j}\, \pounds v\, \pi^\gamma{}_{k]s}. \tag{1.16}$$

In an Fn, if the projective entity $Q^i{}_{hjk}(x, \dot x)$ satisfies the relation:

$$Q^i{}_{hjk((s))} = \mu_s\, Q^i{}_{hjk}, \tag{1.17}$$

where $\mu_s(x)$ is any vector field and then the space under consideration is called special projective recurrent Finsler space or SPR Fn-space and $\mu_s(x)$ is called

3) The indices in $\langle\ \rangle$ are free from symmetric and skew-symmetric operations.

special projective recurrence vector [1]. In view of the equation (1.6) and the last definition we can show that Berwald's curvature tensor will also satisfy the above definition of recurrency.

2. SPECIAL PROJECTIVE MOTION

When an infinitesimal point transformation (1.12) transforms the system of geodesics into the same system, then (1.12) is called an infinitesimal special projective motion. The necessary and sufficient condition that (1.12) be a special projective motion in SPR Fn-space is that the Lie-derivative of $\pi^i_{jk}(x, \dot{x})$ with respect to (1.12) itself has the form :

$$\pounds v\, \pi^i_{jk} = \delta_j^i\, \psi_k + \delta_k^i\, \psi_j \tag{2.1}$$

for a certain non-zero covariant vector $\psi_j(x)$ [8].

Let us introduce a quantity B°_{hj} by the following relation :

$$B^\circ_{hj} \overset{\text{def.}}{=\!=} -\frac{1}{(n^2-1)}\,(n\, Q_{hj} + Q_{hj}). \tag{2.2}$$

By virtue of (1.7), the projective curvature tensor $W^i_{hjk}(x, \dot{x})$ can also be re-written as

$$W^i_{hjk} = Q^i_{hjk} + E^i_{hjk}, \tag{2.3}$$

where

$$E_{h\,k} = -\frac{2}{(n^2-1)}\Big\{(n+1)\, Q_{j|k} - H_{j|k} - H_{[k\langle j\rangle} + (n-1)\, \dot{\partial}_j \dot{\partial}_{[k}\, H -$$
$$- \dot{x}^s\, \dot{\partial}_j\, H^\gamma{}_{\gamma s[k}\Big\}\, \delta^i_{h]}\,. \tag{2.4}$$

We shall also introduce a curvature tensor $W^o_{hjk}(x, \dot{x})$ which will be useful in our theory by the following relation :

$$W^\circ_{hjk}(x, \dot{x}) \overset{\text{def.}}{=\!=} (B^\circ_{hj((k))} - B^\circ_{hk((j))})\,. \tag{2.5}$$

In view of the basic assumption (2.1) and the commutation formula (1.16), we can get

$$\pounds v\, Q^i_{hjk} = \delta_j^i\, \psi_{h((k))} - \delta_k^i\, \psi_{h((j))} + \delta_h^i\, \psi_{j((k))} - \delta_h^i\, \psi_{k((j))}\,, \tag{2.6}$$

where we have used the facts :

$$\text{a)}\quad \pi^i_{hjk}\, \dot{x}^h = 0 \quad\text{and}\quad \text{b)}\quad \psi_s\, \dot{x}^s = 0. \tag{2.7}$$

Contracting the both sides of the formula (2.6) with respect to the indices i and k and making use of (1.5), we obtain

$$\text{£}v\, Q_{hj} = \psi_{j((h))} - n\,\psi_{h((j))}\,. \tag{2.8}$$

Taking the Lie-derivative of the both sides of the formula (2.2) and remembering the above relation, we can find after little simplification :

$$\text{£}v\, B^{\circ}{}_{hj} = \psi_{h((j))}\,. \tag{2.9}$$

Our concerned equations are (2.1) and (2.9), say

$$\text{a)}\quad \text{£}v\, \pi^i{}_{jk} = \delta_j{}^i\,\psi_k + \delta_k{}^i\,\psi_j\,,\quad \text{b)}\quad \text{£}v\, B^{\circ}{}_{jk} = \psi_{j((k))}\,. \tag{2.10}$$

In view of the equations (1.13) and (1.14), the left-hand sides of the above formulae can be rewritten as :

$$\text{a)}\quad \text{£}v\, \pi^i{}_{jk} = v^i{}_{((j))((k))} + Q^i{}_{jkh}\,v^h + \pi^i{}_{sjk}\,v^s{}_{((\gamma))}\,\dot{x}^{\gamma}\ \text{ and }\ \text{b)}\ \text{£}v\, B^{\circ}{}_{jk} = B^{\circ}{}_{jk((h))}\,v^h +$$
$$+ B^{\circ}{}_{hk}\,v^h{}_{((j))} + B^{\circ}{}_{jh}\,v^h{}_{((k))} + \dot{\partial}_h\,B^{\circ}{}_{jk}\,v^h{}_{((s))}\,\dot{x}^s\,. \tag{2.11}$$

The first set of integrability condition of (2.10) is composed of

$$\text{a)}\quad \text{£}v\, W^i{}_{hjk} = 0\,,\quad \text{b)}\quad \text{£}v\, W^{\circ}{}_{hjk} = -\,\psi_s\,W^s{}_{hjk}\,. \tag{2.12}$$

In an affinelly connected SPR Fn-space, let us try to discuss the existence of special projective motion (1.12) satisfying (2.1). For this purpose at first, we have to assume the condition (2.12). In what follows, we shall find an important property on $W^i{}_{hjk}(x, \dot{x})$ holding in SPR Fn-space admitting special projective motion (1.12). From the equations (1.5) and (1.17) we can find

$$Q_{hj((s))} = \mu_s\,Q_{hj}\,. \tag{2.13}$$

Taking the covariant derivative of the both sides of (2.2) with respect to x^s and noting the above formula and the equation (2.2) itself, we get

$$B^{\circ}{}_{hj((s))} = \mu_s\,B^{\circ}{}_{hj}\,. \tag{2.14}$$

By virtue of the equations (1.17), (2.4) and (2.13), we can deduce

$$E^i{}_{hjk((s))} = \mu_s\,E^i{}_{hjk}\,. \tag{2.15}$$

In this way, from the equations (1.17), (2.3) and (2.15), we can get at last an essential property on $W^i{}_{hjk}(x, \dot{x})$ of the form :

$$W^i{}_{hjk((s))} = \mu_s\,W^i{}_{hjk}\,. \tag{2.16}$$

3. CONCRETE FORM OF SPECIAL PROJECTIVE MOTION

Operating the both sides of the formula (2.16) by £v and making use of (2,12a), we have

$$\text{£}v\,(W^i{}_{hjk((s))}) = (\text{£}v\,\mu_s)\,W^i{}_{hjk}\,. \tag{3.1}$$

Applying the commutation formula (1.15) to the projective curvature tensor W^i_{hjk}, we can get

$$\pounds v\,(W^i_{hjk((s))}) - (\pounds v\, W^i_{hjk})_{((s))} = (\pounds v\, \pi^i_{\gamma s})\, W^\gamma_{hjk} - (\pounds v\, \pi^\gamma_{hs})\, W^i_{\gamma jk} - (\pounds v\, \pi^\gamma_{js})\, W^i_{h\gamma k} - - (\pounds v\, \pi^\gamma_{ks})\, W^i_{hj\gamma} - (\pounds v\, \pi^m_{s\gamma})\, \dot{x}^\gamma\, \dot{\partial}_m\, W^i_{hjk}\,. \tag{3.2}$$

Now, introducing the basic formula (2.1) into the right-hand side of the above result and making use of (2.7b), we obtain

$$\pounds v\,(W^i_{hjk((s))}) - (\pounds v\, W^i_{hjk})_{((s))} = \delta_s^i\, \psi_\gamma\, W^\gamma_{hjk} - 2\psi_s\, W^i_{hjk} - \psi_h\, W^i_{sjk} - - \psi_j\, W^i_{hsk} - \psi_k\, W^i_{hjs}\,. \tag{3.3}$$

In view of the equation (2.12a), the last formula takes the form :

$$\pounds v\,(W^i_{hjk((s))}) = -\, 2\psi_s\, W^i_{hjk} + \delta_s^i\, \psi_\gamma\, W^\gamma_{hjk} - \psi_h\, W^i_{sjk} - - \psi_j\, W^i_{hsk} - \psi_k\, W^i_{hjs}\,. \tag{3.4}$$

With the help of the formula (3.1), the above formula takes the form :

$$(2\psi_s + \pounds v\, \mu_s)\, W^i_{hjk} = \delta_s^i\, \psi_\gamma\, W^\gamma_{hjk} - \psi_h\, W^i_{sjk} - \psi_j\, W^i_{hsk} - \psi_k\, W^i_{hjs}\,. \tag{3.5}$$

Contracting the above formula with respect to the indices i and s and remembering the equation (1.11a), we get

$$(2\psi_s + \mu_s)\, W^s_{hjk} = n\, \psi_\gamma\, W^\gamma_{hjk} - \psi_h\, W^s_{sjk} + \psi_j\, W^s_{hks} - \psi_k\, W^s_{hjs}\,. \tag{3.6}$$

By virtue of the equations (1.11b) and (1.11c), the above formula yields :

$$(\pounds v\, \mu_s)\, W^s_{hjk} = (n-2)\, \psi_s\, W^s_{hjk}\,. \tag{3.7}$$

On account of (2.12b), the above relation can be re-written as

$$(\pounds v\, \mu_s)\, W^s_{hjk} = -\,(n-2)\, \pounds v\, W^\circ_{hjk}\,. \tag{3.8}$$

Transvecting the both sides of the formula (3.5) by ψ_i and summing over i, we have

$$(2\,\psi_s + \pounds v\, \mu_s)\, \psi_i\, W^i_{hjk} = \psi_s\, \psi_\gamma\, W^\gamma_{hjk} - \psi_h\, \psi_i\, W^i_{sjk} - - \psi_j\, \psi_i\, W^i_{hsk} - \psi_k\, \psi_i\, W^i_{hjs}\,. \tag{3.9}$$

In order to avoid getting a special form of special projective motion in an SPR Fn-space, we assume here and here after that $\psi_i\, W^i_{hjk}$ does not vanish, say $\pounds v\, W^\circ_{hjk} \neq 0$. In fact, if we have the condition $\psi_i\, W^i_{hjk} = 0$, the vector ψ_i becomes to be restricted by this condition, so the motion is specialized.

After the little simplification the formula (3.9) takes the form :

$$(\pounds v\, \mu_s)\, \psi_i\, W^i_{hjk} = -\, \psi_s\, \psi_i\, W^i_{hjk} - \psi_h\, \psi_i\, W^i_{sjk} - - \psi_j\, \psi_i\, W^i_{hsk} - \psi_k\, \psi_i\, W^i_{hjs}\,. \tag{3.10}$$

For $n \geqslant 3$ the equation (3.7) yields

$$\psi_s W^s{}_{hjk} = \frac{1}{(n-2)} (\pounds v\, \mu_s) W^s{}_{hjk}. \tag{3.11}$$

In this way, introducing the last formula into the both sides of the equation (3.10), and neglecting the number factor $(n-2)$, we can find

$$(\pounds v\, \mu_s)(\pounds v\, \mu_i) W^i{}_{hjk} = -\psi_s (\pounds v\, \mu_i) W^i{}_{hjk} - \psi_h (\pounds v\, \mu_i) W^i{}_{sjk} - \\ - \psi_j (\pounds v\, \mu_i) W^i{}_{hsk} - \psi_k (\pounds v\, \mu_i) W^i{}_{hjs} \tag{3.12}$$

or

$$(\pounds v\, \mu_i)\left[(\pounds v\, \mu_s) W^i{}_{hjk} + \psi_k W^i{}_{hjs} + \psi_s W^i{}_{hjk} + \psi_h W^i{}_{sjk} + \psi_j W^i{}_{hsk}\right] = 0. \tag{3.13}$$

From the equation (3.5), we can deduce

$$\psi_h W^i{}_{sjk} + \psi_j W^i{}_{hsk} + \psi_k W^i{}_{hjs} = \delta_s{}^i \psi_\Upsilon W^\Upsilon{}_{hjk} - (2\psi_s + \pounds v\, \mu_s) W^i{}_{hjk}. \tag{3.14}$$

Substituting the value of the left-hand side of the above formula into the equation (3.13), we get

$$\psi_s W^i{}_{hjk} (\pounds v\, \mu_i) = (\pounds v\, \mu_s) \psi_\Upsilon W^\Upsilon{}_{hjk}. \tag{3.15}$$

By virtue of the equations (2.12b) and (3.8), the above formula takes the form :

$$[(n-2)\psi_s - \pounds v\, \mu_s] \pounds v\, W^\circ{}_{hjk} = 0. \tag{3.16}$$

But, as we have assumed $\pounds v\, W^\circ{}_{hjk} \neq 0$, from (3.16), we can obtain

$$\psi_s = \frac{1}{(n-2)} (\pounds v\, \mu_s) \quad (n \geqslant 3). \tag{3.17}$$

In this way, we can obtain :

Theorem 3.1. If an SPR Fn-space $(n \geqslant 3)$ admits an infinitesimal special projective motion, the motion should be of the form :

$$\bar{x}^i = x^i + v^i(x)\, dt, \ \pounds v\, \pi^i{}_{jk} = \delta_j{}^i \psi_k + \delta_k{}^i \psi_j, \ \psi_k = \frac{1}{(n-2)} (\pounds v\, \mu_k). \tag{3.18}$$

Now, let us examine a case of where $(\pounds v\, \mu_h)$ denotes a parallel vector :

$$(\pounds v\, \mu_h)_{((j))} = 0. \tag{3.19}$$

Introducing the formula (3.17) into the right-hand side of (2.9), we have

$$\pounds v\, B^\circ{}_{hj} = \frac{1}{(n-2)} (\pounds v\, \mu_h)_{((j))}. \tag{3.20}$$

By virtue of the formula (3.19), the above equation takes the form

$$\pounds v\, B^\circ{}_{hj} = 0. \tag{3.21}$$

In view of the equations (2.2), (2.4) and (3.21), we can conclude :

$$\pounds v\, E^i{}_{hjk} = 0. \tag{3.22}$$

Thus, operating the both sides of (2.3) by $\pounds v$ and making use of the last formula, we get

$$\pounds v\, W^i{}_{hjk} = \pounds v\, Q^i{}_{hjk}\,. \tag{3.23}$$

In case of the present theory, we have (2.12a), so the above formula can be rewritten as

$$\pounds v\, Q^i{}_{hjk} = 0. \tag{3.24}$$

But this gives us the parallel (and gradient) property of $(\pounds v\, \mu_k)$. In the following lines we shall try to prove this.

The equations (2.12a) and (3.24) give us (3.22), by which we can find the relation (3.21). In view of (3.21), the formula (2.9) takes the form

$$\psi_{h((j))} = 0. \tag{3.25}$$

With the help of the equations (3.17) and (3.25), we can conclude :

$$(\pounds v\, \mu_h)_{((j))} = 0. \tag{3.26}$$

By virtue of the equations (2.6) and (3.24), we can find

$$(\delta_j{}^i\, \psi_h + \delta_h{}^i\, \psi_j)_{((k))} - (\delta_k{}^i\, \psi_h + \delta_h{}^i\, \psi_k)_{((j))} = 0\,. \tag{3.27}$$

Contracting the above formula with respect to the indices i and h, we have

$$\psi_{j((k))} = \psi_{k((j))}\,. \tag{3.28}$$

On account of equations (3.17) and (3.28), we can get

$$(\pounds v\, \mu_j)_{((k))} = (\pounds v\, \mu_k)_{((j))}\,. \tag{3.29}$$

This completes the proof.

In this way, we have :

Theorem 3.2. When SPR Fn-space admits a general special projective motion, in order that $\pounds v\, \mu_k$ denotes a parallel vector, it is necessary and sufficient that we have $\pounds v\, Q^i{}_{hjk} = 0$.

On the other hand, we know that in order that special projective motion becomes special projective affine motion, it is necessary and sufficient that we have $\psi_h = 0$ or $\pounds v\, \mu_h = 0$, consequently, we get :

Theorem 3.3. In order that a special projective motion admitted in SPR Fn-space becomes a special projective affine motion in the same space, it is necessary and sufficient that we assume $\pounds v\, \mu_h = 0$.

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ÖZET

Bu çalışmada, bir denklem ve bir tanım gözönüne alınarak, Berwald eğrilik tensörünün adı geçen tekrarlılık tanımına uyduğu saptanmaktadır.