PROJECTIVE CURVATURE COLLINEATION IN SYMMETRIC FINSLER SPACE

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In this paper it has been investigated the different cases under which a special conformal motion is a projective curvature collineation.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space Fn [1][1)] equipped with the positively homogeneous metric function $F(x, \dot{x})$ of degree one in its directional arguments. The fundamental metric tensors $g_{ij}(x, \dot{x})$ and $g^{ij}(x, \dot{x})$ are symmetric in their indices and are defined by

$$g_{ij}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \tag{1.1}$$

and

$$g_{ij} g^{ik} = \delta_j^k, \tag{1.2}$$

where δ_j^k are Kronecker deltas. Mishra [2] has defined the projective covariant derivative of a vector field $X^i(x, \dot{x})$ with the help of projective connection parameter $\Pi^i_{hk}(x, \dot{x})$ as follows :

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_h X^i) \Pi^h_{\gamma k} \dot{x}^\gamma + X^h \Pi^i_{hk}, \tag{1.3}$$

where $\Pi^i_{hk}(x, \dot{x})$ is positively homogeneous function being defined by

$$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def.}}{=} G^i_{hk} - \frac{1}{(n+1)} (2\delta^i_{(h} G^\gamma_{k)\gamma} + \dot{x}^i G^\gamma_{\gamma kh})\,^{2)}. \tag{1.4}$$

The following identities hold :

$$\Pi^i_{hk\gamma} \dot{x}^h = \dot{\partial}_h \Pi^i_{k\gamma} \dot{x}^h = 0, \quad \Pi^i_{hk} \dot{x}^h = \Pi^i_k. \tag{1.5}$$

The commutation formula [2] for the projective covariant derivative of a tensor field $T^i_j(x, \dot{x})$ is expressed by

$$2T^i_{j\,[((h))\,((k))]} = T_j^\gamma Q^i_{hk\gamma} - T_\gamma^i Q^\gamma_{hkj} - (\dot{\partial}_\gamma T^i_j) Q^\gamma_{shk} \dot{x}^s, \tag{1.6}$$

where

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1) The numbers in square brackets refer to the references given at the end of the paper.

2) $2A_{(hk)} = A_{hk} + A_{kh}$ and $2A_{[hk]} = A_{hk} - A_{kh}$.

$$Q^i_{jkh}(x, \dot{x}) \overset{\text{def.}}{=} 2\{\dot\partial_{[h} \Pi^i_{k]j} - (\dot\partial_\gamma \Pi^i_{j[k}) \Pi^\gamma_{h]s} \dot{x}^s + \Pi^\gamma_{j[k} \Pi^i_{h]\gamma}\}. \tag{1.7}$$

The projective entities Q^i_{jhk} satisfy the following identities [2]:

$$\text{a) } Q^i_{j(kh)} = 0, \text{ b) } Q^i_{[hjk]} = 0, \text{ c) } Q^i_{jki} = Q_{jk}, \quad \text{d) } Q^i_{jkh} = \dot\partial_j Q^i_{kh}. \tag{1.8}$$

The contractions of Q^i_{jhk} are given by

$$\text{a) } Q_k = Q^i_{ki}, \text{ b) } Q_k = Q_{jk}\dot{x}^j, \text{ c) } \dot\partial_j Q_k = Q_{jk} \text{ and d) } 2Q_{[ij]} = Q^h_{hij}. \tag{1.9}$$

Weyl's curvature tensor can also be written in terms of Berwald's and projective entities as follows [2]

$$W^i_{jkh} = Q^i_{jkh} - \frac{2}{(n^2-1)} \{(n+1) Q_{j[k} - H_{j[k} - H_{[k<j>} +$$

$$+ (n-1)\dot\partial_j \dot\partial_{[k} H - \dot{x}^s \dot\partial_j H^\gamma_{\gamma s[k}\} \delta^i_{h]}{}^{3)}, \tag{1.10}$$

where $H^i_{jhk}(x, \dot{x})$ are Berwald's curvature tensor fields being defined by

$$H^i_{jhk}(x, \dot{x}) = 2\{\partial_{[k} G^i_{h]j} - (G^i_{\gamma j[h}) G^\gamma_{k]s}\dot{x}^s + G^\gamma_{j[h} G^i_{k]\gamma}\} \tag{1.11}$$

and Weyl's curvature tensor field $W^i_{jhk}(x, \dot{x})$ is given by

$$W^i_{jhk}(x, \dot{x}) = H^i_{hjk} + \frac{1}{(n+1)} \{\delta^i_h H^\gamma_{\gamma kj} + \dot{x}^i \dot\partial_h H^\gamma_{\gamma kj} +$$

$$+ 2\delta^i_{[j} (H_{<h>k]} + \dot\partial_{k]} \dot\partial_h H)\}. \tag{1.12}$$

We consider the infinitesimal points transformation

$$\bar{x}^i = x^i + v^i(x)\, dt, \tag{1.13}$$

where $v^i(x)$ is any vector field and dt is an infinitesimal constant. The Lie-derivative of any tensor field $T^i_j(x, \dot{x})$ and the connection parameter Π^i_{jk} is given by

$$\pounds v\, T^i_j(x, \dot{x}) = T^i_{j((h))} v^h - T^h_j v^i_{((h))} + T^i_h v^h_{((j))} + (\dot\partial_h T^i_j) v^h_{((s))} \dot{x}^s \tag{1.14}$$

and

$$\pounds v\, \Pi^i_{jk}(x, \dot{x}) = v^i_{((j))((k))} + Q^i_{hjk} v^h + \Pi^i_{jkh} v^h_{((s))} \dot{x}^s. \tag{1.15}$$

The following commutation formula holds for the operators $\pounds v$ and $((k))$:

$$\pounds v\, T^i_{j((k))} - (\pounds v\, T^i_j)_{((k))} = T^h_j \pounds v\, \Pi^i_{kh} - T^i_h \pounds v\, \Pi^h_{kj} - (\dot\partial_h T^i_j) \pounds v\, \Pi^h_{ks} \dot{x}^s \tag{1.16}$$

and the connection coefficients are related with respect to those operators by

$$2\pounds v\, \Pi^i_{h[k((j))]} = \pounds v\, Q^i_{jkh} + 2(\pounds v\, \Pi^s_{m[j}) \Pi^i_{k]sh} \dot{x}^m. \tag{1.17}$$

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3) The indices in ⟨ ⟩ are free from symmetric and skew symmetric parts.

The conformal transformation in a Finsler space Fn is characterized by

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} , \tag{1.18}$$

where $\sigma = \sigma(x)$ is a scalar point function and $\bar{g}_{ij}$ are the components of a covariant metric tensor in a conformal Finsler space $\overline{Fn}$. In a conformal Finsler space, we have

$$\bar{G}^i (x, \dot{x}) = G^i (x, \dot{x}) - B^{is} \sigma_s , \tag{1.19}$$

which gives

$$\bar{G}_h^i (x, \dot{x}) = G_h^i - B_h^{is} \sigma_s \tag{1.20}$$

$$\bar{G}^i_{hk} (x, \dot{x}) = G^i_{hk} - B^{is}_{hk} \sigma_s \tag{1.21}$$

and

$$\bar{G}^i_{hk\gamma} (x, \dot{x}) = G^i_{hk\gamma} - B^{is}_{hk\gamma} \sigma_s , \tag{1.22}$$

where $B^{hk}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} F^2 g^{hk} - \dot{x}^h \dot{x}^k$ is positively homogeneous of degree two in its directional argument. The function $B^{hk}(x, \dot{x})$ satisfies the following identities :

$$B_k^{is} \overset{\text{def.}}{=} \dot{\partial}_k B^{is} , \; B^{is}_{kh} \overset{\text{def.}}{=} \dot{\partial}_k \dot{\partial}_h B^{is} , \; B^{is}_{hk\gamma} \overset{\text{def.}}{=} \dot{\partial}_k \dot{\partial}_\gamma \dot{\partial}_h B^{is} ,$$
$$B^{is}_{hk\gamma} \dot{x}^\gamma = 0 , \; B^{is}_{hk} \dot{x}^k = B_h^{is} . \tag{1.23}$$

Under the conformal change $\Pi^i_{jk}(x, \dot{x})$ is given by

$$\bar{\Pi}^i_{jk} (x, \dot{x}) = \Pi^i_{jk} \left\{ B^{is}_{jk} - \frac{1}{(n+1)} (2\delta^i_{(j} B^{\gamma s}_{k)\gamma} + \dot{x}^i B^{\gamma s}_{\gamma kj}) \right\} . \tag{1.24}$$

2. PROJECTIVE CURVATURE COLLINEATION

Definition 2.1 (Projective affine motion (Pande and Kumar [4])). An Fn is said to admit a projective affine motion provided there exists a vector field v^i such that

$$\pounds v \, \Pi^i_{jk} = 0. \tag{2.1}$$

Definition 2.2 (Projective curvature collineation (Pande and Kumar [8])). An Fn is said to admit a projective curvature collineation if there exists a vector v^i such that

$$\pounds v \, Q^i_{hjk} = 0. \tag{2.2}$$

Definition 2.3 (Projective Ricci collineation (Pande and Kumar [8])). An Fn is said to admit a Ricci projective curvature collineation if there exists a vector v^i such that

$$\pounds v\, Q_{hk} = 0. \tag{2.3}$$

The variation of $\Pi^i_{jk}(x, \dot{x})$ under the conformal transformation is $\overline{\Pi}^i_{jk}$ and that under infinitesimal point transformation (1.13) is $\pounds v\, \Pi^i_{jk}$. The two transformations will coincide if the corresponding variations are the same. Thus we have :

Theorem 2.1 (Pande and Kumar [7]). A necessary and sufficient condition that the infinitesimal change (1.13) be a special conformal motion is that

$$\pounds v\, \Pi^i_{hk} = -\sigma_s \left\{ B^{is}_{hk} - \frac{1}{(n+1)} (2\delta^i_{(h} B^{\gamma s}_{k)\gamma} + \dot{x}^i B^{\gamma s}_{\gamma kh}) \right\}. \tag{2.4}$$

Thus for a special conformal, we also have

$$\pounds v\, \Pi^i_{hk\gamma} = -\sigma_s \left[B^{is}_{hk\gamma} - \frac{1}{(n+1)} \{2\delta^i_{(h} B^{ns}_{k)n\gamma} + \delta_\gamma{}^i B^{ns}_{nkh} + \right.$$

$$\left. + \dot{x}^i B^{hs}_{nkh\gamma}\} \right]. \tag{2.5}$$

We shall now study the different cases under which a special conformal motion is a projective curvature collineation. Let us suppose that the space admits a special conformal motion then by using equations (1.17) and (2.4), we get

$$\pounds v\, Q^i_{hjk}(x, \dot{x}) = -2 \left[\sigma_{s[((k))} \left\{ B^{is}_{j]h} - \frac{1}{(n+1)} (B^{\gamma s}_{j]\gamma} \delta_h{}^i + \delta_{j]}^i B^{\gamma s}_{\gamma h} + \right.\right.$$

$$\left. + \dot{x}^i B^{\gamma s}_{j]\gamma h}) \right\} + \sigma_s \left\{ B^{is}_{h[j((k))]} - \Pi^i_{\gamma h[j} B^{\gamma s}_{k]\rho} \dot{x}^\rho - \right.$$

$$- \frac{1}{(n+1)} \left\{ \delta_h{}^i B^{\gamma s}_{\gamma[j((k))]} + B^{\gamma s}_{\gamma h[((k))} \delta_{j]}^i + \dot{x}^i B^{\gamma s}_{\gamma h[j((k))]} - \right.$$

$$\left.\left.\left. - \dot{x}^\gamma \Pi^i_{\gamma h[j} B^m_{k]m} \right\} \right\} \right]. \tag{2.6}$$

If the special conformal motion admits a projective curvature collineation then in view of (2.2), the above equation reduces to

$$\sigma_{s[((k))} \left\{ B^{is}_{j]} - \frac{1}{(n+1)} (B^{\gamma s}_{j]\gamma} \dot{x}^i + \delta^i_{j]} B^{\gamma s}_\gamma) \right\} +$$

$$+ \sigma_s \left\{ B^{is}_{[j((k))]} - \frac{1}{(n+1)} (B^{\gamma s}_{\gamma[j((k))]} \dot{x}^i + B^{\gamma s}_{\gamma[((k))} \delta^i_{j]}) \right\} = 0. \tag{2.7}$$

Thus we have the following theorem :

Theorem 2.2. A necessary condition for a special conformai motion to be a projective curvature collineation is that the equation (2.7) holds.

Since the operators of Lie-derivative and the operation of contraction are commutative, therefore, with the help of equations (2.2) and (2.6), we obtain

$$(n+1)\,(\dot{\partial}_s B^{is}{}_j)_{((i))} - B^{\gamma s}{}_{\gamma j}\,\dot{x}^i - \delta_j^i\, B^{\gamma s}{}_{\gamma} - B^{\gamma s}{}_{\gamma((j))} = 0. \tag{2.8}$$

Theorem 2.3. A necessary and sufficient condition that a special conformai motion in a Finsler space be a projective curvature collineation is that (2.8) holds.

3. SPECIAL PROJECTIVE SYMMETRIC FINSLER SPACE

Definition 3.1 (Pande and Kumar [7]). If the entity $Q^i{}_{hjk}(x, \dot{x})$ satisfies the relation

$$Q^i{}_{hjk((\gamma))} = 0, \tag{3.1}$$

then such a space is known as special projective symmetric Finsler space being denoted by Fn^*. The following relations are satisfied in Fn^*:

$$\text{a)}\ Q^i{}_{jk((m))} = 0\,, \quad \text{b)}\ Q^i{}_{j((m))} = 0. \tag{3.2}$$

The commutation formula (1.16) can be written for $Q^i{}_j(x, \dot{x})$ as follows which after using the equation (3.2b) yields

$$-(\pounds v\, Q_j^i)_{((k))} = Q_j^h\, \pounds v\, \Pi^i{}_{kh} - Q_h^i\, \pounds v\, \Pi^h{}_{kj} - (\dot{\partial}_h\, Q_j^i)\, \pounds v\, \Pi^h{}_{ks}\, \dot{x}^s\,. \tag{3.3}$$

Substituting the value of $\pounds v\, \Pi^i{}_{kh}$ from (2.4) in the above equation, we get

$$\begin{aligned}(\pounds v\, Q_j^i)_{((k))} = \sigma_s \Big[& Q_j^h\, B^{is}{}_{kh} - Q_h^i\, B^{hs}{}_{kj} - (\dot{\partial}_h\, Q_j^i)\, B^{hs}{}_{km} - \\ & - \frac{1}{(n+1)} \{Q_j^h\, (\delta_k^i\, B^{\gamma s}{}_{\gamma h} + \delta_h^i\, B^{\gamma s}{}_{\gamma k} + \dot{x}^i\, B^{\gamma s}{}_{\gamma hk}) - Q_h^i\, (\delta_k^h\, B^{\gamma s}{}_{\gamma j} + \\ & + \delta_j^h\, B^{\gamma s}{}_{\gamma k} + \dot{x}^h\, B^{\gamma s}{}_{\gamma jk}) - (\dot{\partial}_h\, Q_j^i)\, (\delta_k^h\, B^{\gamma s}{}_{\gamma m} + \delta_m^h\, B^{\gamma s}{}_{\gamma k} + \\ & + \dot{x}^h\, B^{\gamma s}{}_{\gamma mk})\, \dot{x}^m\} \Big]. \end{aligned} \tag{3.4}$$

Thus, we have the following theorems:

Theorem 3.1. A necessary condition that a projective symmetric space *Fn* admits a special conformai motion is that equation (3.4) holds.

Theorem 3.2. The necessary condition that the projective affine motion is satisfied in *Fn* is that $(\pounds v\, Q_j^i)_{((k))} = 0$ holds. Since a Finsler space is said to be isotropic if

$$W_h^i = 0. \tag{3.5}$$

Therefore from equation (1.10), we have

$$Q^i{}_{jkh} = \frac{2}{(n^2-1)} \Big\{ (n+1)\, Q_{j[k} - H_{j[k} - H_{[k<j>} + \\ + (n-1)\, \dot\partial_j \,\dot\partial_{[k}\, H - \dot x^s \,\dot\partial_j\, H^\gamma{}_{\gamma s[k} \Big\} \,\delta^i{}_{h]}\,. \tag{3.6}$$

Taking Lie-derivative of the above equation and using the fact that the operators £v and $\dot\partial_k$ are commutative, we get

$$£v\, Q^i{}_{jkh} = \frac{2}{(n^2-1)} \Big\{ (n+1)\, £v\, Q_{j[k} - £v\, H_{j[k} - £v\, H_{[k<j>} + \\ + (n-1)\, \dot\partial_j \,\dot\partial_{[k}\, £v\, H - \dot x^s \,\dot\partial_j\, £v\, H^\gamma{}_{\gamma s[k} \Big\} \,\delta^i{}_{h]}\,. \tag{3.7}$$

Now if the space admits a special Ricci collineation (i.e. $£v\, H_{hk} = 0$) and projective Ricci collineation (i.e. $£v\, Q_{hk} = 0$) then from above equation, we get

$$£v\, Q^i{}_{jkh} = 0. \tag{3.8}$$

Thus we have:

Theorem **3.3.** In an isotropic Finsler space every special and projective Ricci collineation is a special curvature collineation.

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Ö Z E T

Bu çalışmada, özel bir konform hareketin bir projektif eğrilik kolineasyonu olduğu çeşitli haller incelenmektedir.