İstanbul Üniv. Fen Fak. Mat. Der., 53 (1994), 29-35

ON A TYPE OF A SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Summary : We define a linear connection on a Riemannian manifold which is semi-symmetric π-recurrent connection and study some properties of the curvature tensor, Ricci tensor and conformal curvature tensor with respect to semi-symmetric π-recurrent connection.

BİR RIEMANN MANİFOLDU ÜZERİNDE YARI SİMETRİK BİR METRİK BAĞLANTI TİPİ HAKKINDA

Özet : Bu çalışmada, bir Riemann manifoldu üzerinde yarı simetrik ve "π-recurrent" olan bir lineer bağlantı tanımlanmakta ve eğrilik tensörü, Ricci tensörü ve konform eğrilik tensörünün yarı simetrik "π-recurrent" bağlantıya göre bazı özelikleri incelenmektedir.

INTRODUCTION

Let M^n be an n-dimensional Riemannian manifold of class C^∞ endowed with a Riemannian metric g and let ∇ be the Levi-Civita connection on M. Let $\bar{\nabla}$ be a linear connection defined on M^n. The torsion tensor $T(X, Y)$ of $\bar{\nabla}$ is given by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \tag{1}$$

where X and Y are arbitrary vector fields. If the torsion tensor T is of the form

$$T(X, Y) = \pi(Y)\, X - \pi(X)\, Y \tag{2}$$

where π is a 1-form, then $\bar{\nabla}$ is called a semi-symmetric connection [1]. The connection $\bar{\nabla}$ is called a metric connection if

$$\bar{\nabla} g = 0. \tag{3}$$

A semi-symemetric connection $\bar{\nabla}$ with torsion tensor $T(X, Y) = \pi(Y)\, X - \pi(X)\, Y$ is defined as a semi-symmetric π-recurrent connection if

$$(\bar{\nabla}_X \pi)\,(Y) = A(X)\, \pi(Y) \tag{4}$$

for arbitrary vector fields X and Y, where A is a non-zero 1-form and Q is a vector field satisfying $g(X, Q) = A(X)$.

The present paper deals with a Riemannian manifold admitting a semi-symmetric π-recurrent connection which is also a metric connection. In Section 1 of the present paper an expression for the curvature tensor $\bar{R}(X, Y)Z$, the Ricci tensor $\bar{S}(Y, Z)$ and the scalar curvature $\bar{r}$ of the connection $\bar{\nabla}$ have been deduced. In Section 2, a necessary and sufficient condition has been deduced for the Ricci tensor of the semi-symmetric metric π-recurrent connection $\bar{\nabla}$ to be symmetric. Also a necessary and sufficient condition has been deduced for the Ricci tensor $\bar{\nabla}$ to be skew-symmetric. Further, it is shown that the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is symmetric if and only if the curvature tensor $\bar{R}$ of $\bar{\nabla}$ satisfies first Bianchi identity. In Section 3, it is shown that the conformal curvature tensors for $\bar{\nabla}$ and ∇ are equal. Also it is shown that if the curvature tensor of $\bar{\nabla}$ vanishes then the manifold is conformally flat.

1. **Preliminaries.** It is known [2] that for a semi-symmetric metric connection $\bar{\nabla}$

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y) X - g(X, Y) P \tag{1.1}$$

where P is a vector field defined by $g(X, P) = \pi(X)$ for every vector field X. Let $\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z$ and

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

be the curvature tensors of the connections $\bar{\nabla}$ and ∇ respectively. Then by virtue of (1.1) we get

$$\begin{aligned}\bar{R}(X, Y)Z = {} & \bar{\nabla}_X \{\nabla_Y Z + \pi(Z) Y - g(Y, Z) P\} - \bar{\nabla}_Y \{\nabla_X Z + \pi(Z) X - \\ & - g(X, Z) P\} - \{\nabla_{[X, Y]} Z + \pi(Z)[X, Y] - g([X, Y], Z) P\}\end{aligned}$$

or

$$\begin{aligned}\bar{R}(X, Y)Z = {} & \nabla_X \nabla_Y Z + \pi(\nabla_Y Z) X - g(X, \nabla_Y Z) P + X\pi(Z) Y + \\ & + \pi(Z)\{\nabla_X Y + \pi(Y) X - g(X, Y) P\} - \bar{\nabla}_X g(Y, Z) P - \\ & - g(Y, Z)\{\nabla_X P + \pi(P) X - g(X, P) P\} - \nabla_Y \nabla_X Z - \pi(\nabla_X Z) Y + \\ & + g(Y, \nabla_X Z) P - Y\pi(Z) X - \pi(Z)\{\nabla_Y X + \pi(X) Y - \\ & - g(Y, X) P\} + \bar{\nabla}_Y g(X. Z) P + g(X, Z)\{\nabla_Y P + \pi(P) Y - \\ & - g(Y, P) P\} - \nabla_{[X, Y]} Z - \pi(Z)[X, Y] + g([X, Y], Z) P\end{aligned}$$

or

$$\begin{aligned}\bar{R}(X, Y) Z - R(X, Y) Z + \{(\nabla_X \pi) (Z) Y - \pi(Z) \pi(X) Y + \\ + \pi(P) g(X, Z) Y\} - \{(\nabla_Y \pi)(Z) X + \pi(Z) \pi(Y) X - \\ - \pi(P) g(Y, Z) X\} - g(Y, Z)(\nabla_X P) + g(X, Z)(\nabla_Y P) + \\ + g(Y, Z) \pi(X) P - g(X, Z) \pi(Y) P.\end{aligned} \tag{1.2}$$

From (4) and (1.1) it follows that

$$(\nabla_X \pi)(Y) - \pi(X) \pi(Y) + \pi(P) g(X, Y) = A(X) \pi(Y). \tag{1.3}$$

From (1.1) we also get

$$\bar{\nabla}_X P - \pi(P) X = \nabla_X P - \pi(X) P. \tag{1.4}$$

Since $(\bar{\nabla}_X g)(Y, Z) = 0$, we get $(\nabla_X g)(P, Z) = 0$

or

$$\bar{\nabla}_X \pi(Z) - \pi(\bar{\nabla}_X Z) - g(\bar{\nabla}_X P, Z) = 0$$

or

$$(\bar{\nabla}_X \pi) Z - g(\bar{\nabla}_X P, Z) = 0$$

or

$$A(X) \pi(Z) = g(\bar{\nabla}_X P, Z)$$

or

$$A(X) g(P, Z) = g(\bar{\nabla}_X P, Z)$$

i.e.,

$$\bar{\nabla}_X P = A(X) P. \tag{1.5}$$

From (1.4) and (1.5) it follows that

$$A(X) P - \pi(P) X = \nabla_X P - \pi(X) P. \tag{1.6}$$

By virtue of (1.3) and (1.6) we get from (1.2)

$$\begin{aligned}\bar{R}(X, Y) Z = R(X, Y) Z + \pi(P) \{g(Y, Z) X - g(X, Z) Y\} + \\ + A(X) \{\pi(Z) Y - g(Y, Z) P\} - \\ - A(Y) \{\pi(Z) X - g(X, Z) P\}.\end{aligned} \tag{1.7}$$

Then from (1.7) we have

$$\begin{aligned}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + \\ + \pi(P) \{g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\} + \\ + A(X) \{\pi(Z) g(Y, W) - g(Y, Z) g(P, W)\} - \\ - A(Y) [\pi(Z) g(X, W) - g(X, Z) g(P, W)]\end{aligned} \tag{1.8}$$

where

$$R(X, Y, Z, W) = g(R(X, Y) Z, W).$$

Let $\bar{S}(X, Y)$ and $S(X, Y)$ be the Ricci tensors of the connections $\bar{\nabla}$ and ∇ respectively. Also let $\bar{r}$ and r be the scalar curvature of the connections $\bar{\nabla}$ and ∇ respectively. Putting $X = W = e_i$ in (1.8) where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at a point and 1 is summed for $1 \leqq i \leqq n$ we get

$$\bar{S}(Y, Z) = S(Y, Z) + (n-1)\pi(P) g(Y, Z) - (n-2)\pi(Z) A(Y) - g(Y, Z) A(P). \tag{1.9}$$

Again putting $Y = Z = e_i$ in (1.9) we get

$$\bar{r} = r + n(n-1)\pi(P) - 2(n-1) A(P). \tag{1.10}$$

Thus the curvature tensor, Ricci tensor and the scalar curvature of $\bar{\nabla}$ is given by (1.8), (1.9) and (1.10) respectively.

2. By virtue of (1.9) we get

$$\bar{S}(Z, Y) = S(Z, Y) + (n-1)\pi(P) g(Z, Y) - (n-2)\pi(Y) A(Z) - g(Z, Y) A(P). \tag{2.1}$$

From (1.9) and (2.1) we get

$$\bar{S}(Y, Z) - \bar{S}(Z, Y) = (n-2)[\pi(Y) A(Z) - \pi(Z) A(Y)]. \tag{2.2}$$

If $\bar{S}(X, Y)$ is symmetric the left hand side of (2.2) vanishes and we get

$$\pi(Y) A(Z) = \pi(Z) A(Y) \quad (n \geqq 3). \tag{2.3}$$

Hence we can state the following theorem:

Theorem 1. A necessary and sufficient condition for the Ricci tensor of the semi-symmetric metric π-recurrent connection $\bar{\nabla}$ to be symmetric is that the relation (2.3) holds.

Again, for $n=2$, from (2.2) we have $\bar{S}(X, Y)$ is symmetric. This leads to the following theorem:

Theorem 2. If a Riemannian manifold of dimension 2 admits a semi-symmetric metric π-recurrent connection $\bar{\nabla}$ then the Ricci tensor of $\bar{\nabla}$ is symmetric.

From (1.9) and (2.1) we get

$$\bar{S}(Y, Z) + \bar{S}(Z, Y) = 2S(Y, Z) + 2g(Y, Z)[(n-1)\pi(P) - A(P)] - (n-2)[\pi(Z) A(Y) + \pi(Y) A(Z)]. \tag{2.4}$$

If $\bar{S}(X, Y)$ is skew-symmetric the left hand side of (2.4) vanishes and we get

$$S(Y, Z) = \frac{(n-2)}{2} [\pi(Z) A(Y) + \pi(Y) A(Z)] - \\ - g(Y, Z) [(n-1) \pi(P) - A(P)]. \tag{2.5}$$

On the other hand, if $S(Y, Z)$ is given by (2.5), then from (2.4) we get

$$\bar{S}(Y, Z) + \bar{S}(Z, Y) = 0.$$

Thus we have the following theorem :

Theorem 3. If a Riemannian manifold of dimension $n (n \geqq 3)$ admits a semi-symmetric metric π-recurrent connection $\bar{\nabla}$, then a necessary and sufficient condition for the Ricci tensor of $\bar{\nabla}$ to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection ∇ is given by (2.5).

From (1.8) we have

$$\bar{R}(X, Y, Z, W) + \bar{R}(Y, X, Z, W) = 0. \tag{2.6}$$

Using (1.8) and the first Bianchi identity with respect to the Levi-Civita connection, we have

$$\begin{aligned} \bar{R}(X, Y, Z, W) + \bar{R}(Y, Z, X, W) + \bar{R}(Z, X, Y, W) = \\ = g(Y, W) [A(X) \pi(Z) - A(Z) \pi(X)] + \\ + g(Z, W) [A(Y) \pi(X) - A(X) \pi(Y)] + \\ + g(X, W) [A(Z) \pi(Y) - A(Y) \pi(Z)]. \end{aligned} \tag{2.7}$$

We call (2.7) as the first Bianchi identity with respect to semi-symmetric π-recurrent metric connection $\bar{\nabla}$.

In particular, if the Ricci tensor $\bar{S}$ is symmetric then (2.7) reduces to

$$\bar{R}(X, Y, Z, W) + \bar{R}(Y, Z, X, W) + \bar{R}(Z, X, Y, W) = 0. \tag{2.8}$$

Hence from (2.7) and Theorem 1 we get

Theorem 4. The Ricci tensor of M^n with respect to the semi-symmetric π-recurrent metric connection $\bar{\nabla}$ is symmetric if and only if the condition (2.8) holds.

3. **Conformal curvature tensor.** Let $\bar{C}(X, Y, Z, U)$ and $C(X, Y, Z, U)$ be the covariant conformal curvature tensors of the connections $\bar{\nabla}$ and ∇ respectively. Then

$$\begin{aligned} \bar{C}(X, Y, Z, U) = \bar{R}(X, Y, Z, U) - \frac{1}{n-2} [\bar{S}(Y, Z) g(X, U) - \\ - \bar{S}(X, Z) g(Y, U) + g(Y, Z) \bar{S}(X, U) - g(X, Z) \bar{S}(Y, U)] + \\ + \frac{\bar{r}}{(n-1)(n-2)} [g(Y, Z) g(X, U) - g(X, Z) g(Y, U)]. \end{aligned} \tag{3.1}$$

Applying (1.8), (1.9) and (1.10) in (3.1) it follows that

$$\bar{C}(X, Y, Z, U) = C(X, Y, Z, U). \tag{3.2}$$

Thus we have the following theorem :

Theorem 5. If a Riemannian manifold admits a semi-symmetric metric π-recurrent connection then its conformal curvature tensor is same as the conformal curvature tensor of the manifold.

Now suppose that the Ricci tensor of the semi-symmetric metric π-recurrent connection $\bar{\nabla}$ vanishes.

That is,

$$\bar{S}(X, Y) = 0. \tag{3.3}$$

Hence

$$\bar{r} = 0. \tag{3.4}$$

Applying (3,3) and (3.4) in (3.1) we get

$$\bar{C}(X, Y, Z, U) = \bar{R}(X, Y, Z, U). \tag{3.5}$$

Hence from Theorem 5 and (3.5) it follows that

$$C(X, Y, Z, U) = \bar{R}(X, Y, Z, U). \tag{3.6}$$

Thus we have the following theorem :

Theorem 6. If a Riemannian manifold admits a semi-symmetric metric π-recurrent connection $\bar{\nabla}$ whose Ricci tensor vanishes, then the curvature tensor of the connection $\bar{\nabla}$ is qual to the conformal curvature tensor of the manifold.

If the curvature tensor of the semi-symmetric metric π-recurrent connection $\bar{\nabla}$ vanishes, then the Ricci tensor also vanishes. From (3.6) we have

$$C(X, Y, Z, U) = \bar{R}(X, Y, Z, U).$$

But by hypothesis

$$\bar{R}(X, Y, Z, U) = 0.$$

Therefore,

$$C(X, Y, Z, U) = 0.$$

Hence we can state the following corollary :

Corollary. If a Riemannian manifold admits a semi-symmetric metric π-recurrent connection whose curvature tensor vanishes, then the manifold is conformally flat.

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