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ON A TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Summary : In a recent paper [1] Agashe and Chafle have studied a semi-symmetric non-metric connection on a Riemannian manifold. In the present paper we have studied a semi-symmetric non-metric connection whose torsion tensor is recurrent.

BİR RIEMANN MANİFOLDU ÜZERİNDE YARI SİMETRİK VE METRİK OLMAYAN BİR BAĞLANTI TİPİ HAKKINDA

Özet : Agashe ve Chafle, 1992 deki bir çalışmalarında, bir Riemann manifoldu üzerinde yarı simetrik ve metrik olmayan bir bağlantıyı incelemişlerdi. Bu çalışmada ise burulma tensörü "recurrent" olan ve metrik olmayan bir yarı simetrik bağlantı incelenmektedir.

1. Introduction. Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with a metric tensor g. A linear connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection [1], if its torsion tensor T satisfies

$$T(X, Y) = \pi(Y) X - \pi(X) Y \tag{1}$$

and

$$(\bar{\nabla}_X g)(Y, Z) = -\pi(Y) g(X, Z) - \pi(Z) g(X, Y) \tag{2}$$

for arbitrary vector fields X, Y and Z in M^n, where π is a 1-form defined by $g(X, V) = \pi(X)$.

We now suppose that the Riemannian manifold (M^n, g) admits a semi-symmetric non-metric connection $\bar{\nabla}$ given by

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y) X \tag{3}$$

for arbitrary vector fields X and Y in M^n, where ∇ denotes the Levi-Civita connection.

Further, if $\bar{R}$ and R denote the curvature tensor of $\bar{\nabla}$ and ∇ respectively, then it is known [1] that

$$\bar{R}(X, Y) Z = R(X, Y) Z + M(X, Z) Y - M(Y, Z) X \tag{4}$$

where M is a tensor field of type (0,2) defined by

$$M(X, Y) = (\nabla_X \pi)(Y) - \pi(X) \pi(Y). \tag{5}$$

In the present paper we have considered a Riemannian manifold (M^n, g) admitting a type of semi-symmetric non-metric connection whose torsion tensor is recurrent. Here we have deduced a necessary and sufficient condition for the symmetry of the Ricci tensor of the semi-symmetric non-metric connection whose torsion tensor is recurrent. In Section 3 it is shown that if the curvature tensor of the semi-symmetric non-metric connection vanishes then the associate vector field V is a Ricci principal direction corresponding to the eigen value $(n-1)\, a\, \pi(V)$ and the manifold is projectively flat and conformally flat. Further, it is shown that if a Riemannian manifold (M^n, g) is a group manifold with respect to the semi-symmetric non-metric connection then (M^n, g) is flat.

2. A special type of semi-symmetric non-metric connection. We consider a type of semi-symmetric non-metric connection $\bar{\nabla}$ whose torsion tensor T satisfies the following condition

$$(\bar{\nabla}_X T)(Y, Z) = B(X) T(Y, Z) \tag{2.1}$$

where B is a non-zero 1-form.

From (1) we have

$$(C_1^1 T)(Y) = (n-1) \pi(Y). \tag{2.2}$$

From (2.2) we get

$$(\bar{\nabla}_X C_1^1 T)(Y) = (n-1)(\bar{\nabla}_X \pi)(Y). \tag{2.3}$$

Now from (2.1) and (2.2) we have

$$(\bar{\nabla}_X C_1^1 T)(Y) = B(X)(C_1^1 T)(Y) = (n-1) B(X) \pi(Y). \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$(\bar{\nabla}_X \pi)(Y) = B(X) \pi(Y). \tag{2.5}$$

From (3) we get

$$(\bar{\nabla}_X \pi)(Y) = (\nabla_X \pi)(Y) - \pi(X) \pi(Y). \tag{2.6}$$

Thus from (2.5) and (2.6) we have

$$(\nabla_X \pi)(Y) = \pi(X) \pi(Y) + B(X) \pi(Y). \tag{2.7}$$

From (5) and (2.7) we get

$$M(X, Y) = B(X) \pi(Y). \tag{2.8}$$

Using (2.8) we can write (4) as follows:

$$\bar{R}(X, Y)Z = R(X, Y)Z + [B(X)Y - B(Y)X]\pi(Z) \tag{2.9}$$

From (2.9) we have the following theorem:

Theorem 1. If a Riemannian manifold admits a semi-symmetric non-metric connection whose torsion tensor T satisfies the condition (2.1), then the curvature tensor of the manifold is given by (2.9).

Now from (2.9) we get

$$\bar{S}(Y, Z) = S(Y, Z) - (n-1)B(Y)\pi(Z) \tag{2.10}$$

where $\bar{S}$ and S denote the Ricci tensor corresponding to the semi-symmetric non-metric connection $\bar{\nabla}$ and Levi-Civita connection ∇ respectively.

From (2.10) we get

$$\bar{S}(Y, Z) - \bar{S}(Z, Y) = (n-1)[B(Z)\pi(Y) - B(Y)\pi(Z)]. \tag{2.11}$$

Thus we get the theorem:

Theorem 2. The Ricci tensor of the semi-symmetric non-metric connection with recurrent torsion tensor is symmetric if and only if $B(Y)\pi(Z) = B(Z)\pi(Y)$.

Next we shall find another necessary and sufficient condition for which the Ricci tensor for the semi-symmetric non-metric connection $\bar{\nabla}$ will be symmetric.

Now from (2.9) and the first Bianchi identity with respect to the Riemannian connection, we get

$$\begin{aligned}\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = {} & [B(X)\pi(Z) - B(Z)\pi(X)]Y + \\ & + [B(Y)\pi(X) - B(X)\pi(Y)]Z + \\ & + [B(Z)\pi(Y) - B(Y)\pi(Z)]X.\end{aligned} \tag{2.12}$$

Hence from (2.12) and Theorem 2, we can state the following theorem:

Theorem 3. The Ricci tensor of the semi-symmetric non-metric connection with recurrent torsion tensor is symmetric if and only if

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

3. In this section we consider

$$\bar{R}(X, Y)Z = 0 \tag{3.1}$$

and

$$(\bar{\nabla}_X T)(Y, Z) = B(X)T(Y, Z). \tag{3.2}$$

From (2.9) we get

$$R(X, Y) Z = [B(Y) X - B(X) Y] \pi(Z). \tag{3.3}$$

From (3.3) we have

$$S(Y, Z) = (n - 1) B(Y) \pi(Z). \tag{3.4}$$

From the symmetric relation of the Ricci tensor it is found that $B(Y) = a \pi(Y)$ where $B(V) = a \pi(V)$, $\pi(V) \neq 0$. Thus (3.3) and (3.4) take the form

$$R(X, Y) Z = a [\pi(Y) X - \pi(X) Y] \pi(Z) \tag{3.5}$$

and

$$S(Y, Z) = a(n - 1) \pi(Y) \pi(Z). \tag{3.6}$$

Let P be the symmetric linear operator such that

$$S(X, Y) = g(PX, Y). \tag{3.7}$$

Thus from (3.6) and (3.7), we get

$$PV = a(n - 1) \pi(V) V. \tag{3.8}$$

Hence we have the following theorem:

Theorem 4. If a Riemannian manifold admits a semi-symmetric non-metric connection $\bar{\nabla}$ with V, associated vector field and curvature tensor $\bar{R}$ and the torsion tensor T satisfy the conditions (3.1) and (3.2), then

(i) the curvature tensor R and Ricci tensor S of the manifold (M^n, g) are respectively given by (3.5) and (3.6),

(ii) the associated vector field V is a Ricci principal direction with corresponding eigen value $(n - 1) a \pi(V)$.

Let $\bar{W}$ and W denote the Weyl projective curvature tensor of $\bar{\nabla}$ and ∇ respectively. Then it can be easily seen that

$$\bar{W}(X, Y) Z = W(X, Y) Z. \tag{3.9}$$

From (3.1) and (3.9) we get

$$W(X, Y) Z = 0. \tag{3.10}$$

It is known that a Riemannian manifold is a space of constant curvature if and only if it is projectively flat and a Riemannian manifold of constant curvature is conformally flat [2].

Hence from (3.10) we can state the following theorem:

Theorem 5. If (M^n, g) is a Riemannian manifold with vanishing curvature tensor with respect to the semi-symmetric non-metric connection, then (M^n, g) is a space of constant curvature and hence is conformally flat.

A Riemannian manifold (M^n, g) is gruop manifold [3] with respect to the semi-symmetric non-metric connection if

$$\bar{R}(X, Y) Z = 0 \tag{3.11}$$

and

$$(\bar{\nabla}_X T)(Y, Z) = 0\,. \tag{3.12}$$

Hence from (3.5), we get

$$R(X, Y) Z = 0\,. \tag{3.13}$$

Hence we can state following theorem:

Theorem 6. If a Riemannian manifold admits a semi-symmetric non-metric connection for which the manifold is group manifold, then (M^n, g) is flat and consequently (M^n, g) is projectively flat and conformally flat.

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