# ON THE $\mathscr{A}$-CONTINUITY OF REAL FUNCTIONS

E. SAVAŞ - G. DAS
Department of Mathematics, Yüzüncü Yıl University, Van-TURKEY
Department of Mathematics, Utkal University, Bhubaneswar, Orissa-INDIA

**Summary** : In [1] J. Antoni and T. Salat defined the concept of $A$-continuity for real functions. In [9] it was introduced some new concepts of continuity for real functions and studied the relations between different concepts of continuity. In the present paper authors have generalized this concept of continuity of real functions by taking a sequence of matrices $A^i = (a_{nk}(i))$ of complex numbers and have taken some new results.

## REEL FONKSİYONLARIN $\mathscr{A}$-SÜREKLİLİĞİ HAKKINDA

**Özet** : Bu çalışmada, 1980 yılında J. Antoni ve T. Salat tarafından reel fonksiyonlar için tanımlanmış olan $A$-süreklilik kavramı, kompleks elemanlı bir $A^i = (a_{nk}(i))$ matris dizisi alınarak genelleştirilmekte ve bazı yeni sonuçlar elde edilmektedir.

## 1. INTRODUCTION

Let $l_\infty$ and $c$ respectively denote the Banach spaces of bounded and convergent sequences $x \quad (x_n)$ with the usual norm $\| x \| = \sup_n | x_n |$. A sequence $x \in l_\infty$ is almost convergent (see Lorentz, [3]). Banach limits of $x$ coincide. Let $\hat{c}$ denote the space of almost convergent sequences (see Lorentz, [3]).

Let $\sigma$ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^p(n) \neq n$ for all positive integers $n$ and $p$ where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$, $p=1, 2, \ldots$ . A continuous linear functional $\varphi$ on the space of complex bounded sequences is an invariant mean or a $\sigma$-mean, if $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$, $\varphi(1, 1, 1, \ldots) = 1$ and $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all bounded sequences $x$.

Let $V_\sigma$ be the set of all bounded sequences all of whose invariant means are equal (see, Schaefer [10]). In case $\sigma(n) = n+1$, the $\sigma$-means are classical Banach limits on $l_\infty$ and $V_\sigma$ is the set of almost convergent sequences (see, Lorentz [3]).

If $x = (x_n)$, write $Tx = (x_{\sigma(n)})$ . It is easy to show that

$$V_\sigma = \{x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) = L, \text{ uniformly in } n, \ L = \sigma\text{-lim } x\}$$

where

$$t_{mn}(x) = (x_n + Tx_n + \dots + T^m x_n) / (m+1)$$

and

$$t_{-1,n}(x) = 0.$$

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ be a sequence of real numbers. The sequence $(A(x)_n)$ defined by

$$A(x)_n = \sum_{k=1}^{\infty} a_{nk} x_k \tag{1}$$

is called the $A$-transform of $x$ whenever the above series converges for $n = 1, 2, \dots$. The sequence $x$ is said to be $A$-summable to $x_0$ if the sequence $(A(x)_n)$ converges to $x_0$. $A$ is called conservative if $x \in c$ implies $(A(x)_n) \in c$. $A$ is called regular if it is conservative and preserves the limit of each convergent sequence.

## 2. PRELIMINARIES

A function $f$: $\mathbf{R} \to \mathbf{R}$ is called $C$-continuous at a point $x_0 \in \mathbf{R}$, if $(C,1)$-$\lim f(x_n) = f(x_0)$ whenever $(C,1)$-$\lim x_n = x_0$ (see, problem 4216 [8]) where $(C,1)$ is the first Cesàro mean and $(C,1)$-$\lim x_n = x_0$ means that

$$\frac{x_1 + x_2 + \dots + x_n}{n} \to x_0 \quad (n \to \infty). \tag{2}$$

In the paper [7] the following result is proved:

Let $A$ be a regular matrix and $f: \mathbf{R} \to \mathbf{R}$ a function. If $A$-$\lim f(x_n)$ exists in $\mathbf{R}$ whenever $(x_n)$ converges, then $f$ is a continuous function on $\mathbf{R}$.

In connection with the result from [8] the following question arises : Is it possible to generalize the mentioned result by replacing the Cesàro matrix by the another regular matrix? The answer of this question was given by Antoni and Salat [1].

They defined the $A$-continuity of real functions in the following manner: Let $A$ be a regular matrix. We shall say that the function $f$: $\mathbf{R} \to \mathbf{R}$ is $A$-continuous at the point $x_0 \in \mathbf{R}$, if the following statament is true :

$$\text{A-}\lim x_n = x_0 \Rightarrow A\text{-}\lim f(x_n) = f(x_0).$$

Now we are ready to begin.

Let $\mathscr{A}$ denote the sequence of matrices $A^i = (a_{nk}(i))$ of real numbers. We write for any sequence $x = (x_n)$, $A_n^i(x) = \sum_{k=1}^{\infty} a_{nk}(i) x_k$ if it exists for each $n$ and $i$. We write

$$A^i(x) = (A_n^i(x))_{n=1}^{\infty}, \quad \mathscr{A}x = (A^i(x))_{i=1}^{\infty}.$$

We define the identity matrix $I$ by

$$a_{nk}(i) = 1 \ (n = k) \text{ for all } i, = 0 \ (n \neq k)$$

for all $i$, so that $\mathscr{A}x = x$ in the case $\mathscr{A} = I$. A sequence $x$ is said to be summable by the method $(\mathscr{A})$ to the complex number $s$ if $\lim_n A_n^i(x) = s$, uniformly in $i$. The method $(\mathscr{A})$ is said to be conservative, if the $\mathscr{A}$-transform of $(x_n)$ is convergent uniformly in $i$, when $(x_n) \in c$. It is called regular if the $\mathscr{A}$-transform of $(x_n)$ is convergent uniformly in $i$ to the limit of $(x_n)$ for each $x = (x_n) \in c$.

3. We shall now establish the following definitions and theorems:

Before giving the main characterization, we need to state a few definitions.

**Definition 1.** Let $(\mathscr{A})$ be a regular method and $x = (x_n)$ be a sequence in $\mathbf{R}$. We shall say that the function $f: \mathbf{R} \to \mathbf{R}$ is $\mathscr{A}$-continuous at the point $x_0 \in \mathbf{R}$ if the following statament is true:

$$\mathscr{A}\text{-lim } x_n = x_0 \Rightarrow \mathscr{A}\text{-lim } f(x_n) = f(x_0).$$

**Definition 2.** Let $(\mathscr{A})$ be a regular method. We shall say that the method $(\mathscr{A})$ has the property $L(a)$ if there exists a sequence $(\eta_k)$, $\eta_k = 0$ or $1$ $(k = 1, 2, \ldots)$ for which $\mathscr{A}$-lim $\eta_k = a$.

In the case $a_{nk}(i) = \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n), k)$, $\mathscr{A}$-continuity reduces to invariant $A$-continuity (see, Savaş [9]), if $a_{nk}(i) = \frac{1}{i+1} \sum_{j=n}^{n+i} a_{jk}$, then $\mathscr{A}$-contiunuity reduces to almost $A$-continuity (see, Öztürk [6]). Of course, where $a_{nk}(i) = a_{nk}$ for all $i$, then $\mathscr{A}$-continuity reduces to $A$-continuity (see, [1]).

In [2] Buck has defined that a sequence is convergent if there is a regular method which sums each of its subsequences. Therefore, replacing the ordinary convergence by the so called $\mathscr{A}$-convergence we obtain the following corollary:

**Corollary 1.** If a function $f: \mathbf{R} \to \mathbf{R}$ is $\mathscr{A}$-continuous at $x_0 \in \mathbf{R}$, then $f$ is continuous at the same point.

The solution of Cauchy functional equation says that if a function is additive and is continuous at least at one point then it is a linear function. Following the above result we record the following observation:

**Theorem 1.** Let $(\mathscr{A})$ be a regular method with property $L(a)$ for a number $a$, $a \neq 0,1$ and let $f : \mathbf{R} \to \mathbf{R}$ be an $\mathscr{A}$-continuous function at each point of $\mathbf{R}$. Then $f$ is a linear function.

The proof is easy and we omit it.

We can also write that if $f$ is continuous and linear then $f$ is of the form $f(x) = \lambda x$ for some real number $\lambda$ (see, Mehdi [5]).

**Theorem 2.** Let $(\mathscr{A})$ be a regular method and $f : \mathbf{R} \to \mathbf{R}$ has the following property: there exists such a point $x_0 \in \mathbf{R}$ that the following implication

$$\mathscr{A}\text{-}\lim x_n = x_0 \Longrightarrow f(x_n) \to f(x_0) \tag{3}$$

is valid. Then

(a) $f$ is a continuous function;

if further the method $(\mathscr{A})$ has the property $L(a)$, $a \neq 0,1$, then

(b) $f$ is a linear function;

if further the method $(\mathscr{A})$ is translative, then

(c) $f$ is a constant function.

**Proof.** (a) Since $\mathscr{A}$ is regular

$$x_n \to x_0 \Longrightarrow \mathscr{A}\text{-}\lim x_n = x_0 .$$

Hence, we get from (3) that

$$x_n \to x_0 \Longrightarrow f(x_n) \to f(x_0).$$

(b) The assertion (3) implies the following:

$$\mathscr{A}\text{-}\lim x_n = x_0 \Longrightarrow \mathscr{A}\text{-}\lim f(x_n) = f(x_0) .$$

Since $\mathscr{A}$ is regular and has the property $L(a)$, we get from Theorem 1 that $f$ is linear.

(c) Let $a, b \in \mathbf{R}$ $(a \neq b)$ and consider the sequence $(x_n) = (a, b, a, b, \ldots)$. Since $\mathscr{A}$ has the property $L(a)$ and is translative, we get

$$a + b = (a + b) \lim_n \sum_{k=1}^{\infty} a_{nk}(i), \text{ uniformly in } i$$

$$= \lim_n \sum_{k=1}^{\infty} a_{nk}(i)(x_k + x_{k+1}), \text{ uniformly in } i$$

$$= \lim_n \sum_{k=1}^{\infty} a_{nk}(i)\, x_k + \lim_n \sum_{k=1}^{\infty} a_{nk}(i)\, x_{k+1}, \text{ uniformly in } i$$

$$= 2 \lim_n \sum_{k=1}^{\infty} a_{nk}(i)\, x_k, \text{ uniformly in } i.$$

Hence,

$$\mathscr{A}\text{-}\lim x_n = \frac{a+b}{2} = x_0 \text{ (say)}.$$

Therefore by assumption

$$(f(x_n)) = (f(a), f(b), f(a), f(b), \dots) \to f\left(\frac{a+b}{2}\right) = f(x_0),$$

and this is not possible unless

$$f(a) = f(b). \tag{4}$$

Hence, it follows from (4) and from (b) that $f$ is $a$ constant function.

If we take

$$a_{nk}(i) = \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n), k)$$

and

$$a_{nk}(i) = \frac{1}{i+1} \sum_{j=n}^{n+i} a_{jk}$$

in the above results respectively, we get the following results:

Note that $f: \mathbb{R} \to \mathbf{R}$ is called invariant $A$-continuous at $x_0 \in \mathbb{R}$ if $\sigma$-lim $A(f(x)) = f(x_0)$ whenever $\sigma$-lim $(Ax) = x_0$ (see, Savaş [9]).

**Corollary 2.** Let $A$ be a regular matrix and $f: \mathbb{R} \to \mathbf{R}$ be an invariant $A$-continuous function at a point $x_0 \in \mathbf{R}$. Then $f$ is continuous at the point $x_0$.

**Corollary 3.** Let $A$ be $a$ regular matrix with property $L(a)$ for a number $a$, $a \neq 0,1$ and let $f: \mathbb{R} \to \mathbf{R}$ be an invariant $A$-continuous function at aeach point of $\mathbf{R}$. Then $f$ is a linear function.

**Corollary 4.** Let $A$ be a regular matrix and $f: \mathbf{R} \to \mathbf{R}$ be an almost $A$-continuous function at a point $x_0 \in \mathbf{R}$. Then $f$ is continuous at the point $x_0$.

**Corollary 5.** Let $A$ be a regular matrix with property $L(a)$ for a number $a$, $a \neq 0,1$ and let $f: \mathbf{R} \to \mathbf{R}$ be an almost $A$-continuous function at each point of $\mathbf{R}$. Then $f$ is a linear function.

## REFERENCES

[1] ANTONI, J. - SALAT, T. : *On the A-continuity of real functions,* Acta F.R.N. Univ. Com. Mathematice, **XXXIX** (1980), 159-164.

[2] BUCK, R.C. : *An addendum to "A note on subsequences",* Proc. Amer. Math. Soc., 7 (1956), 1074-1075.

[3] LORENTZ, G.G. : *A contribution to the theory of divergent sequences,* Acta Math., **80** (1948), 167-190.

[4] MADDOX, I.J. : **Elements of Functional Analysis,** Camb. Univ. Press, London, 1970.

[5] MEHDI, M.R. : *On convex functions,* Jour. London Math. Soc., 39 (1964), 321-326.

[6] ÖZTÜRK, E. : *On almost A-continuity and almost A-continuity of real functions,* Comm. Fac. Sci. Univ. Ankara, Ser. $A_1$ (1983), 25-30.

[7] POSNER, E.C. : *Summability-preserving functions,* Proc. Amer. Math. Soc., **12** (1961), 73-76.

[8] Problem 4216 (1946,470), Amer. Math. Monthly, Proposed by H. Robins, Solution by R.C. Buck, Amer. Math. Monthly, **55** (1948), 36.

[9] SAVAŞ, E. : *On invariant continuity and invariant A-continuity of real functions,* Jour. Orissa Math. Soc., 3 (1984), No. 2, 25.

[10] SCHAEFER, P. : *Infinite matrices and invariant means,* Proc. Amer. Math. Soc., (36), (1) (1972), 104-110.

[11] STIEGLITZ, M. : *Eine Verallgemeinerung* des *Begriffs der Fastkonvergenz,* Math. Japan, **18** (1973), 53-70.