ON A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Summary : In this paper a particular type of semi-symmetric metric connection on a Riemannian manifold satisfying certain condition on the Ricci tensor has been studied.

BİR RIEMANN MANİFOLDU ÜZERİNDE YARI SİMETRİK BİR METRİK BAĞLANTI TİPİ HAKKINDA

Özet : Bu çalışmada, Ricci tensörü belirli bir koşulu gerçekleyen bir Riemann manifoldu üzerinde özel bir yarı simetrik metrik bağlantı tipi incelenmektedir.

INTRODUCTION

Friedmann and Schouten [1] introduced semi-symmetric connection. Yano [2] synthesized the notion of semi-symmetric connection and a metric connection with torsion [3]. He also showed that a Riemannian manifold admits a semi-symmetric metric connection of zero curvature tensor if and only if it is conformally flat [2]. The object of this paper is to study a Riemannian manifold which admits a semi-symmetric metric connection with a certain form of Ricci-tensor.

Consider an n-dimensional oricntable Riemannian manifold with a metric tensor g and its Levi-Civita connection ∇. We consider all geometric objects on M be sufficiently smooth. Denote arbitrary vector fields on M by X, Y and Z. A linear connection $\overline{\nabla}$ on M is said to be semi-symmetric metric connection [2] if there exists a 1-form π such that the torsion tensor T is given by

$$T(X, Y) = \pi(Y)\, X - \pi(X)\, Y \tag{1}$$

and

$$\overline{\nabla} g = 0 .$$

For such a metric connection [2]

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y) X - g(X, Y) \xi \tag{2}$$

1980 Mathematics subject classification (AMS) : Primary 53C05

where ξ is a vector field such that $g(\xi, X) = \pi(X)$. We denote the curvature tensor, Ricci tensor of type (0, 2), the scalar curvature and the Weyl conformal curvature tensor of M with respect to ∇ by K, S, r, C respectively. A bar over them refers to $\overline{\nabla}$. We know that [2]

$$\bar{K}(X, Y) Z = K(X, Y) Z - \alpha(Y, Z) X + \alpha(X, Z) Y - \\ - g(Y, Z) AX + g(X, Z) AY \tag{3}$$

where

$$\alpha(X, Y) = (\nabla_X \pi) Y - \pi(X) \pi(Y) + \left(\frac{1}{2}\right) \pi(\xi) g(X, Y) \tag{4}$$

$$AX = \nabla_X \pi - \pi(X) \xi + \left(\frac{1}{2}\right) \pi(\xi) X. \tag{5}$$

In this connection we recall that $S(X, Y) = \sum_{i=1}^{n} K(X, V_i, V_i, Y)$ where $\{V_i\}$ is an orthonormal basis of the tangent space at each point of the manifold M.

$$C(X, Y) Z = K(X, Y) Z + l(Y, Z) X - l(X, Z) Y + \\ + g(Y, Z) LX - g(X, Z) LY \tag{6}$$

where

$$l(X, Y) = -\frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y) \tag{7}$$

and

$$g(LX, Y) = l(X, Y). \tag{8}$$

1. In this section we deal with the implication of the prescription $\bar{S} = \varphi S + \psi g$, where φ and ψ are real functions on M.

Lemma 1. If $\bar{S} = \varphi S + \psi g$, where φ, ψ are stated earlier then

$$(1 - \varphi) S(X, Y) = (n - 2) \alpha(X, Y) + (a + \psi) g(X, Y) \tag{A}$$

$$(1 - \varphi) r = 2(n - 1) \operatorname{div} \xi + (n - 1)(n - 2) \pi(\xi) + n \psi \tag{B}$$

$$\alpha(X, Y) = (\varphi - 1) l(X, Y) - \frac{\psi}{2(n - 1)} g(X, Y) \tag{C}$$

$$\nabla_X \xi = \pi(X) \xi - \left(\frac{1}{2}\right) \pi(\xi) X - (1 - \varphi) LX - \frac{\psi}{2(n - 1)} X \tag{D}$$

where a is the trace of A and $n \geq 3$.

Proof. From (3) we can write

$$g[\bar{K}(X, Y) Z, W] = g[K(X, Y) Z, W] - g[\alpha(Y, Z) X, W] + \\ + g[\alpha(X, Z) Y, W] - g[g(Y, Z) AX, W] + \\ + g[g(X, Z) AY, W]. \tag{1.1}$$

Putting $X = W$ in (1.1) we get

$$g[\bar{k}(X, Y) Z, X] = g[k(X, Z) Y, X] - g[\alpha(Y, Z) X, X] + \\ + g[\alpha(X, Z) Y, X] - g[g(Y, Z) AX, X] + \\ + g[g(X, Z) AY, X]. \tag{1.2}$$

Let us take $X = V_i$, then (1.2) becomes

$$g[\bar{K}(V_i, Y) Z, V_i] = g[K(V_i, Y) Z, V_i] - g[\alpha(Y, Z) V_i, V_i] + \\ + g[\alpha(V_i, Z) Y, V_i] - g[g(Y, Z) AV_i, V_i] + \\ + g[g[V_i, Z) AY, V_i]. \tag{1.3}$$

Hence from (1.3) we get

$$\bar{S}(Y, Z) = S(Y, Z) - n\,\alpha(Y, Z) + 2\alpha(Y, Z) - ag(Y, Z). \tag{1.4}$$

Now from the given hypothesis we have

$$(1 - \varphi) S(Y, Z) = (n - 2)\,\alpha(Y, Z) + (a + \psi)\, g(Y, Z). \tag{1.5}$$

This completes the proof of (A).

Using the relation (4) we get from (A)

$$(1 - \varphi) S(X, Y) = \\ = (n - 2)\left[(\nabla_X \pi)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(\xi)\, g(X, Y)\right] + \\ + (a + \psi)\, g(X, Y) \\ = (n - 2)\left[g(Y, \nabla_X \xi) - g(X, \xi)\, g(Y, \xi) + \frac{1}{2} g(\xi, \xi)\, g(X, Y)\right] + \\ + \left(\operatorname{div} \xi + \frac{n-2}{2}\pi(\xi)\right) g(X, Y) + \psi\, g(X, Y). \tag{1.6}$$

Putting $X = Y = V_i$ in (1.6) we get

$$(1 - \varphi)\, r = 2(n - 1) \operatorname{div} \xi + (n - 1)(n - 2)\, \pi(\xi) + n\psi.$$

This completes the proof of (B).

Putting $X = Y = V_i$ in (A) gives

$$(1 - \varphi)\, r = (n - 2)\, a + (a + \psi)\, n. \tag{1.7}$$

Using (A) and (1.7) in (7) we get

$$l(X, Y) = -\frac{1}{n-2}\left[\frac{1}{1-\varphi}\{(n-2)\,\alpha(X, Y) + (a+\psi)\,g(X, Y)\}\right] + \\ + \frac{\frac{1}{1-\varphi}\{(n-2)\,a + (a+\psi)\,n\}}{2(n-1)(n-2)}\,g(X, Y). \tag{1.8}$$

Now from (1.8) we have

$$\alpha(X, Y) = (\varphi - 1)\,l(X, Y) - \frac{\psi}{2(n-1)}\,g(X, Y).$$

This completes the proof of (C).

From (8) we can write

$$g((\varphi - 1)\,LX, \xi) = (\varphi - 1)\,l(X, \xi). \tag{1.9}$$

Now using (C) we get from (1.9)

$$g((\varphi - 1)\,LX, \xi) = \alpha(X, \xi) + \frac{\psi}{2(n-1)}\,g(X, \xi). \tag{1.10}$$

Using (4) in (1.10) we have

$$\pi((\varphi-1)\,LX) = (\nabla_X \pi)\,\xi - \pi(X)\,\pi(\xi) + \frac{1}{2}\,\pi(\xi)\,\pi(X) + \frac{\psi}{2(n-1)}\,\pi(X). \tag{1.11}$$

In virtue of (1.11) we can write

$$(\varphi - 1)\,LX = \nabla_X \xi - \pi(X)\,\xi + \frac{1}{2}\,\pi(\xi)\,X + \frac{\psi}{2(n-1)}\,X,$$

i.e.,

$$\nabla_X \xi = \pi(X)\,\xi - \frac{1}{2}\,\pi(\xi)\,X - (1 - \varphi)\,LX - \frac{\psi}{2(n-1)}\,X.$$

This completes the proof of (D).

Now using (C) in (3) we get

$$\bar{K}(X, Y)\,Z = K(X, Y)\,Z - \left[(\varphi - 1)\,l(Y, Z) - \frac{\psi}{2(n-1)}\,g(Y, Z)\right]X + \\ + \left[(\varphi - 1)\,l(X, Z) - \frac{\psi}{2(n-1)}\,g(X, Z)\right]Y - \\ - g(Y, Z)\,AX + g(X, Z)\,AY. \tag{1.12}$$

Appropriate use of (5) in (1.12) gives us

$$\bar{K}(X, Y)Z = \varphi K(X, Y) Z + (1 - \varphi) C(X, Y) Z + \\ + \frac{\psi}{2(n-1)} [g(Y, Z) X - g(X, Z) Y]. \tag{1.13}$$

Thus we can state the following theorem:

Theorem 1.1. If a Riemannian manifold admits a semi-symmetric metric connection such that $\bar{S} = \varphi S + \psi g$, φ and ψ are real functions on M, then for $n > 3$ the following relation holds:

$$\bar{K}(X, Y) Z = \varphi K(X, Y) Z + (1 - \varphi) C(X, Y) Z + \\ + \frac{\psi}{2(n-1)} [g(Y, Z) X - g(X, Z) Y].$$

If $\varphi = 0$, $\psi = 0$, from Theorem 1.1 we have the following generalized version of the result of Imai [4] and therefore Yano [2]:

Corollary 1.1. If a Riemannian manifold admits a semi-symmetric metric connection with vanishing Ricci tensor, then the curvature tensor of the semi-symmetric metric connection is equal to the Weyl conformal curvature tensor.

2. Integrating (B) over M and using Green's theorem [5] we get

$$\int_M \left[\left\{ (1 - \varphi) \frac{r}{n-1} \right\} - (n-2) g(\xi, \xi) - \frac{n}{n-1} \psi \right] dv = 0. \tag{2.1}$$

Let $\bar{S} = S$ ($\varphi = 1$, $\psi = 0$) or $r = 0$, $\psi = 0$, then

$$\int_M g(\xi, \xi) \, dv = 0 \tag{2.2}$$

which implies $g(\xi, \xi) = 0$ and hence $\xi = 0$, as g is positive definite. Now $\xi = 0$ would mean $\bar{\nabla} = \nabla$ and hence ∇ would not be semi-symmetric.

Thus we can state the following theorem :

Theorem 2.1. If M is a compact orientable Riemannian manifold without boundary then neither $\bar{S}$ is identically equal to S nor r and ψ both vanish.

REFERENCES

[1] FRIEDMANN, A. and SCHOUTEN, J.A. : *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Zeitschr., 21 (1924), 211-223.

[2] YANO, K. : *On semi-symmetric metric connection*, Rev. Roum. Math. Pures et Appl., **15** (1970), 1579-1581.

[3] HAYDEN, H.A. : *Subspaces of space with torsion*, Proc. London Math. Soc., **34** (1932), 27-50.

[4] IMAI, T. : *Notes on semi-symmetric metric connection*, Tensor, N.S., **24** (1972), 293-296.

[5] YANO, K. : **Integral formulas in Riemannian Geometry**, Marcel Dekker Inc., N.Y., 1970.