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ON P-SASAKIAN MANIFOLD

M. TARAFDAR - A. MAYRA

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Calcutta-700019, INDIA

Summary : A type of P-Sasakian manifold in which $R(\xi, X) . C = 0$ has been considered, where R is the curvature transformation and C is the conformal curvature tensor of the manifold. It has been shown that such a manifold is conformally flat and hence is an SP-Sasakian manifold.

P-SASAKIAN MANIFOLD HAKKINDA

Özet : Bu çalışmada, $R(\xi, x) . C=0$ koşuluna uyan bir tür P-Sasakian manifold incelenmekte (burada R eğrilik transformasyonunu ve C, manifoldun konform eğrilik tensörünü göstermektedir) ve böyle bir manifoldun konform olarak düz ve dolayısıyla bir SP-Sasakian manifold olduğu gösterilmektedir.

1. Introduction. In a recent paper [3] U.C. De and N. Guha proved that if in a P-Sasakian manifold (M^n, g) $(n > 3)$ the relation $R(x, y) . C = 0$ holds, where $R(x, y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors, X, Y and C is the conformal curvature tensor, then the manifold is conformally flat and hence is an SP-Sasakian manifold. We have generalized this result by taking the weaker hypothesis $R(\xi, x) . C = 0$ instead of $R(x, y) . C = 0$ in a P-Sasakian manifold.

2. Preliminaries. Let (M^n, g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the conditions

$$(\nabla_x \eta) y - (\nabla_y \eta) x = 0 \tag{2.1}$$

$$(\nabla_x \nabla_y \eta) y = -g(x, z) \eta(y) - g(x, y) \eta(z) + 2 \eta(x) \eta(y) \eta(z) \tag{2.2}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M^n, g) admits a vector field ξ and a $(1-1)$ tensor field ϕ such that

$$g(x, \xi) = \eta(x) \tag{2.3}$$

$$\eta(\xi) = 1 \tag{2.4}$$

$$\nabla_x \xi = 1 \tag{2.5}$$

then such a manifold is called para Sasakian manifold or briefly a P-Sasakian manifold by I. Sato and K.Matsumoto [4], [5]. This paper deals with a type of of P-Sasakian manifold in which

$$R(\xi, x) \, . \, C = 0 \tag{2.6}$$

where C is the conformal curvature tensor, R is the Riemannian curvature and $R(x, y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors x, y.

Let (M^n, g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the condition

$$(\nabla_x \eta)\, y = -g(x, y) + \eta(x)\eta(y)\,. \tag{2.7}$$

If moreover (M^n, g) admits a vector field ξ and a (1,1) tensor field ϕ such that conditions (2.3), (2.4), (2.5) are satisfied, then it can be verified that a (M^n, g) becomes a P-Sasakian manifold. Such a P-Sasakian manifold has been called a special P-Sasakian manifold or briefly a SP-Sasakian manifold by Sato and Matsumoto [4], [5].

It is known [4], [5] that in a P-Sasakian manifold the following relations hold:

$$\phi\,\xi = 0 \tag{2.8}$$

$$\phi^2 x = x - \eta(x)\,\xi \tag{2.9}$$

$$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y) \tag{2.10}$$

$$S(x, \xi) = -(n-1)\,\eta(x) \tag{2.11}$$

$$\eta(R(x, y)z) = g(x, z)\,\eta(y) - g(y, z)\,\eta(x) \tag{2.12}$$

$$R(\xi, x)\,y = \eta(y)\,x - g(x, y)\,\xi \tag{2.13}$$

$$R(\xi, x)\,\xi = x - \eta(x)\,\xi \tag{2.14}$$

$$R(x, y)\,\xi = \eta(x)\,y - \eta(y)\,x. \tag{2.15}$$

We shall use these formulas later on.

3. P-Sasakian manifold satisfying $R(\xi, x)\,.\,C = 0$. The conformal curvature tensor C is given by

$$C(x, y)\,z = R(x, y)\,z - \\ - \frac{1}{n-2}\Big[g(y, z)\,Qx - g(x, z)\,Qy + S(y, z)\,x - S(x, z)\,y\Big] + \\ + \frac{r}{(n-1)(n-2)}\Big[g(y, z)\,x - g(x, z)\,y\Big] \tag{3.1}$$

where S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [3], i.e.

$$g(Q\,x, y) = S(x, y).$$

Therefore, $\eta(C(x, y) z) = g(C(x, y) z, \xi)$

or

$$\eta(C(x, y) z) = \frac{1}{n-2}\left[\left(\frac{r}{n-1}+1\right)\{g(y, z)\,\eta(x) - g(x, z)\,\eta(y)\} - \right.$$
$$\left. - \{S(y, z)\,\eta(x) - S(x, z)\,\eta(y)\}\right]. \tag{3.2}$$

Taking $z = \xi$ in (3.2) we get

$$\eta(C(x, y)\,\xi) = 0. \tag{3.3}$$

Again, taking $x = \xi$ in (3.2) we find

$$\eta(C(\xi, y) z) = \frac{1}{n-2}\left[\left(1+\frac{r}{n-1}\right) g(y, z) - S(y, z) - \left(\frac{r}{n-1}+n\right)\eta(y)\,\eta(z)\right]. \tag{3.4}$$

Now

$$(R(\xi, x) \,.\, C)(u, v)\,w = R(\xi, x)\,C(u, v)\,w - C(R(\xi, x)\,u, v)\,w -$$
$$- C(U, R(\xi, x)\,v)\,w - C(u, v)\,R(\xi, x)\,w\,.$$

Using (2.6) we get

$$R(\xi, x)\,C(u, v)\,w - C(R(\xi, x)\,u, v)\,w - C(u, R(\xi, x)\,v)\,w -$$
$$- C(u, v)\,R(\xi, x)\,w = 0\,. \tag{3.5}$$

or

$$g(R(\xi, x)\,C(u, v)\,w, \xi) - g(C(R(\xi, x)\,u, v)\,w, \xi) -$$
$$- g(C(u, R(\xi, x)\,v, w), \xi) - g(C(u, v)\,R(\xi, x)\,w, \xi) = 0\,. \tag{3.6}$$

From this, it follows that

$$C(u, v, w, x) - \eta(x)\,\eta(C(u, v)\,w) + \eta(u)\,\eta(C(x, v)\,w) +$$
$$+ \eta(v)\,\eta(C(u, x)\,w) + \eta(w)\,\eta(C(u, v)\,x) - g(x, u)\,\eta(C(\xi, v)\,w) - \tag{3.7}$$
$$- g(x, v)\,\eta(C(u, \xi)\,w) - g(x, w)\,\eta(C(u, v)\,\xi) = 0\,.$$

where $C(u, v, w, x) = g(C(u, v)\,w, x)$. Taking $x = u$ in (3.7) we get

$$C(u, v, w, x) + \eta(v)\,\eta(C(u, x)\,w) + \eta(w)\,\eta(C(u, v)\,x -$$
$$- g(u, u)\,\eta(C(\xi, v)\,w) - g(u, v)\,\eta(C(u, \xi)\,w) - g(u, w)\,\eta(C(u, v)\,\xi) = 0. \tag{3.8}$$

Let $e_i : i = 1,2,\ldots, n$ be an orthonormal basis of the tangent space at each point of the manifold. Then the sum for $u = e_i$, $1 \leqslant i \leqslant n$ gives

$$\eta\,(C\,(\xi, v)\,w) = 0\,. \tag{3.9}$$

Using (3.3) we find from (3.7)

$$\begin{aligned} &C\,(u, v, w, x) - \eta\,(x)\,\eta\,(C\,(u, v)\,w) + \eta\,(u)\,\eta\,(C\,(x, v)\,w) + \\ &+ \eta\,(v)\,\eta\,(C\,(u, x)\,w) + \eta\,(w)\,\eta\,(C\,(u, v)\,x) - g\,(x, u)\,\eta\,(C\,(\xi, v)\,w) - \\ &- g\,(x, v)\,\eta\,(C\,(u, \xi)\,w) = 0\,. \end{aligned} \tag{3.10}$$

In virtue of (3.4) and (3.9) we find

$$S\,(y,z) = \left(\frac{r}{n-1} + 1\right) g\,(y,z) - \left(\frac{r}{n-1} + n\right) \eta\,(y)\,\eta\,(z). \tag{3.11}$$

Using (3.2), (3.9) and (3.11) the relation (3.10) reduces to

$$C\,(u, v, w, x) = 0\,, \tag{3.12}$$

i.e.

$$C\,(u, v)\,w = 0\,. \tag{3.13}$$

Thus we can state

Theorem. If a P-Sasakian manifold (M^n, g) $(n > 3)$ satisfies tehe relation $R\,(\xi, x)\,.\; C = 0$, then the manifold is conformally flat and hence is an SP-Sasakian manifold [1].

The above theorem has been proved by U.C. De and N. Guha [3].

REFERENCES

[1] ADATI, T. and MIYAZAWA, T. : *Some properties of P-Sasakian manifolds,* TRU Math., **13** (1977), 33-42.

[2] BISHOP, R.L. and GOLDBERG, S.I. : *On conformally flat spaces with commuting curvature and Ricci transformations,* Canad. J. Math., **XXIV** (1972), No. 5, 799-804.

[3] DE, U. C. and GUHA, N. : *On a type of P-Sasakian manifold,* İstanbul Üniv. Fen Fak. Mat. Der., **51** (1992), 35-39.

[4] SATO, I. : *On a structure similar to the almost contact structure,* Tensor, N.S., **30** (1976), 219-224.

[5] SATO, I. and MATSUMOTO, K. : *On P-Sasakian manifolds satisfying certain conditions,* Tensor, N.S., **33** (1979), 173-178.