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SIMULTANEOUS APPROXIMATION BY MODIFIED BETA OPERATORS

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Summary : We define a new type of positive linear operators by combining Beta operators and Lupas operators to approximate integrable functions on $[0, \infty)$ and study the rate of convergence in simultaneous approximation.

MODİFİYE EDİLMİŞ BETA OPERATÖRLERİ İLE EŞZAMANLI YAKLAŞIM

Özet : Bu çalışmada, $[0, \infty)$ üzerinde integre edilebilen fonksiyonlara yaklaşım için, Beta operatörleri ile Lupas operatörleri birleştirilerek, yeni bir pozitif lineer operatör tipi tanımlanmaktadır.

1. INTRODUCTION

To approximate integrable functions on $[0, \infty)$, we modify Beta operators by taking the weight functions of Lupas operators as

$$B_n(f, x) = \frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t)\, dt, \quad x \in [0, \infty) \tag{1.1}$$

where

$$b_{n,k}(x) = \frac{1}{B(k+1, n)} \frac{x^k}{(1+x)^{n+k+1}}, \quad p_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}},$$

$B(k+1, n)$ being the Beta function given by $k!\,(n-1)!\,/\,(n+k)!$.

By $\mathscr{L}$, we shall denote the class of all Lebesque integrable functions on $[0, \infty)$ satisfying

$$\int_0^{\infty} \frac{|f(t)|}{(1+t)^n}\, dt < \infty \text{ for some positive integer } n.$$

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The above class of functions $\mathscr{L}$ is bigger than the class of all Lebesque integrable functions on $[0, \infty)$.

In this paper we obtain a Voronovskaja type asymptotic formula and an error estimation for the operators (1.1).

2. BASIC RESULTS

In this section, we mention some basic results in the form of lemmas, which will be used in the sequel.

Lemma 2.1. For $m \in \mathbf{N} \cup \{0\}$, if

$$U_{n,m}(x) = \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \left(\frac{k}{n+1} - x\right)^m$$

then there exists the following recurrence relation

$$(n+1)\, U_{n,m+1}(x) = x(1+x)\,[U'_{n,m}(x) + m\; U_{n,m-1}(x)].$$

Consequently

(i) $U_{m\,m}(x)$ is a polynomial in x of degree at most m.

(ii) $U_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

The proof of the above lemma is simple and left for the readers.

Lemma 2.2 [2]. There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that

$$\frac{d^r}{dx^r}(x^k(1+x)^{-n-k}) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^k (k-nx)^j\, q_{i,j,r}(x)\, x^{k-r}(1+x)^{-n-k-r}.$$

The proof of the above lemma easily follows also along the lines given in [1].

Lemma 2.3. For $r \in \mathbf{N} \cup \{0\}$, if we define

$$T_{r,n,m}(x) = \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} b_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t)\,(t-x)^m\, dt$$

then

$$T_{r,n,0}(x) = 1,\; T_{r,n,1}(x) = \frac{x(2r+3)+r+1}{n-r-2} \tag{2.1}$$

and there holds

$$(n-m-r-2)\, T_{r,n,m+1}(x) = x(1+x)\,[T'_{r,n,m}(x) + 2m\, T_{r,n,m-1}(x)] + \\ + [(1+2x)(m+r+1)+x]\, T_{r,n,m}(x),\ n>m+r+2. \tag{2.2}$$

Proof. The proof of (2.1) easily follows from the definition of $T_{r,n,m}(x)$. To prove (2.2) we proceed as follows:

$$x(1+x)\, b'_{n,k}(x) = [k-(n+1)x]\, b_{n,k}(x)$$

and

$$t(1+t)\, p'_{n,k}(t) = (k-nt)\, p_{n,k}(t).$$

We have from the definition of $T_{r,n,m}(x)$

$$x(1+x)\, T'_{r,n,m}(x) = \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} x(1+x)\, b'_{n+r,k}(x) \cdot \int_0^{\infty} p_{n-r,k+r}(t)\,(t-x)^m\, dt - mx(1+x)\, T_{r,n,m-1}(x).$$

Therefore

$$x(i+x)\,[T'_{r,n,m}(x) + m\, T_{r,n,m-1}(x)] =$$

$$= \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} [k-(n+r+1)x]\, b_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t)\,(t-x)^m\, dt -$$

$$- \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} b_{n+r,k}(x) \int_0^{\infty} [[k+r-(n-r)\,t] + (n-r)(t-x) - \\ -[r(1+2x)+x]]\, p_{n-r,k+r}(t)\,(t-x)^m\, dt =$$

$$= \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} b_{n+r,k}(x) \int_0^{\infty} t(1+t)\, p'_{n-r,k+r}(t)\,(t-x)^m\, dt + \\ +(n-r)\, T_{r,n,m+1}(x) - [r(1+2x)+x]\, T_{r,n,m}(x) =$$

$$= \frac{n-r-1}{n+r} \sum_{k=0}^{\infty} b_{n+r,k}(x) \int_0^{\infty} [(1+2x)(t-x) + (t-x)^2 + \\ + x(1+x)]\, p_{n-r,k+r}(t)\,(t-x)^m\, dt + (n-r)\, T_{r,n,m+1}(x) - \\ -[r(1+2x)+x]\, T_{r,n,m}(x) = \\ = -(m+1)(1+2x)\, T_{r,n,m}(x) - \\ -(m+2)\, T_{r,n,m+1}(x) - mx(1+x)\, T_{r,n,m-1}(x) + \\ +(n-r)\, T_{r,n,m+1}(x) - [r(1+2x)+x]\, T_{r,n,m}(x).$$

Hence, we get the required result (2.2).

From this recurrence relation, we have

$$T_{r,n,2}(x)=\frac{x^2[(2n-1)+(2r+5)(2r+3)]+x[(2n-1)+(2r+5)(r+1)+ +(2r+3)(r+2)]+(r+2)(r+1)}{(n-r-2)(n-r-3)}.$$

Lemma 2.4. If f is r times $(r=1,2,\ldots)$ differentiable on $[0,\infty)$, then we have

$$B_n^{(r)}(f,x)=\frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}\sum_{k=0}^{\infty}b_{n+r,k}(x)\int_0^{\infty}p_{n-r,k+r}(t)f^{(r)}(t)\,dt.$$

Proof. We have

$$B_n^{(r)}(f,x)=\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}^{(r)}(x)\int_0^{\infty}p_{n,k}(t)f(t)\,dt.$$

By using Leibnitz theorem, we obtain

$$B_n^{(r)}(f,x)=\frac{n-1}{n}\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(n+k+r-i)!}{(n-1)!\,(k-i)!}(-1)^{r-i}x^{k-i}(1+x)^{-n-k-1-r-i}$$

$$\cdot\int_0^{\infty}p_{n,k}(t)f(t)\,dt$$

$$=\frac{n-1}{n}\sum_{k=0}^{\infty}\frac{(n+k+r)!}{(n-1)!\,k!}\,\frac{x^k}{(1+x)^{n+k+r+1}}$$

$$\cdot\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}p_{n,k+i}(t)f(t)\,dt$$

$$=\frac{(n-1)\,(n+r-1)!}{n!}\sum_{k=0}^{\infty}b_{n+r,k}(x)$$

$$\cdot\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}p_{n,k+i}(t)f(t)\,dt.$$

Again, by using Leibnitz theorem, we get

$$p^{(r)}_{n-r,\,k+r}(t)=\frac{(n-1)\,!}{(n-r-1)\,!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}p_{n,\,k+i}(t).$$

Hence,

$$B^{(r)}_n(f,x)=\frac{(n-r-1)\,!\,(n+r-1)\,!}{(n-2)\,!\,n\,!}\sum_{k=0}^{\infty}b_{n+r,\,k}(x)\int_0^{\infty}(-1)^r p^{(r)}_{n-r,\,k+r}(t)\,f(t)\,dt$$

integrating r times by parts, we get the required result.

3. MAIN RESULTS

Theorem 3.1. Let $f\in\mathscr{L}$ be bounded on every finite subinterval of $[0,\infty)$. If $f^{(r+2)}$ exists at a fixed point $x\in(0,\infty)$ and $f(t)=O(t^{\alpha})$ as $t\to\infty$ for some $\alpha>0$, then we have

$$\lim_{n\to\infty} n\,[B^{(r)}_n(f,x)-f^{(r)}(x)]=r(r+2)f^{(r)}(x)+[x(2r+3)+r+1]f^{(r+1)}(x)+$$
$$+x(1+x)f^{(r+2)}(x).$$

Proof. By Taylor's expansion of f we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i\,!}(t-x)^i+\varepsilon(t,x)\,(t-x)^{r+2} \tag{3.1}$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x)=O((t-x)^{\alpha})$ as $t\to\infty$ for some $\alpha>0$.

Using (3.1) in Lemma 2.4, we have

$$n\,[B^{(r)}_n(f,x)-f^{(r)}(x)]=$$

$$=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i\,!}B^{(r)}_n((t-x)^i,x)+n\,B^{(r)}_n(\varepsilon(t,x)\,(t-x)^{r+2},x)-n\,f^{(r)}(x)=$$

$$=\frac{(n-r-2)\,!\,(n+r)\,!}{n\,!\,(n-2)\,!}\,n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)\,(n-r-1)}{i\,!\,(n+r)}\sum_{k=0}^{\infty}b_{n+r,\,k}(x)\right.$$
$$\left.\cdot\int_0^{\infty}p_{n-r,\,k+r}(t)\,\frac{d^r}{dx^r}(t-x)^i\,dt-f^{(r)}(x)\right]+$$

$$+n\left[\frac{(n-r-2)\,!\,(n+r)\,!}{n\,!\,(n-2)\,!}-1\right]f^{(r)}(x)+$$

$$+n\,\frac{(n-1)}{n}\sum_{k=0}^{\infty} b_{n,k}^{(r)}(x)\int_0^{\infty} p_{n,k}(t)\,\varepsilon(t,x)\,(t-x)^{r+2}\,dt=$$

$$=n\left[\left\{\frac{(n-r-2)\,!\,(n+r)\,!}{n\,!\,(n-2)\,!}-1\right\}f^{(r)}(x)+\frac{x\,(2r+3)+r+1}{n-r-2}\,f^{(r+1)}(x)+\right.$$

$$\left.+\left\{\frac{[(2n-1)+(2r+5)(2r+3)]\,x^2+[(2n-1)+(2r+5)(r+1)+(2r+3)(r+2)]\,x+(r+2)(r+1)}{(n-r-2)(n-r-3)}\right\}\frac{f^{(r+2)}(x)}{2\,!}\right]+$$

$$+E_{n,r}(x),$$

by Lemma 2.3, where

$$E_{n,r}(x)=n\sum_{k=0}^{\infty} b_{n,k}^{(r)}(x)\int_0^{\infty} p_{n,k}(t)\,\varepsilon(t,x)\,(t-x)^{r+2}\,dt.$$

To prove the theorem, it is sufficient to show that

$$I_n=(n-1)\sum_{k=0}^{\infty} b_{n,k}^{(r)}(x)\,x^r(1+x)^r\int_0^{\infty} p_{n,k}(t)\,\varepsilon(t,x)\,(t-x)^{r+2}\,dt$$

tends to zero as $n\to\infty$.

Using Lemma 2.2, we get

$$|I_n|\leqslant(n-1)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,\,j\geq 0}}(n+1)^i\,|[k-(n+1)\,x]|^j\,|q_{i,j,r}(x)|\,b_{n,k}(x)$$

$$\cdot\int_0^{\infty} p_{n,k}(t)\,|\varepsilon(t,x)|\;|t-x|^{r+2}\,dt\leqslant$$

$$\leqslant(n-1)\,K(x)\sum_{\substack{2i+j\leq r\\ i,\,j\geq 0}}(n+1)^i\sum_{k=0}^{\infty} b_{n,k}(x)\,|[k-(n+1)\,x]|^j$$

$$\cdot\int_0^{\infty} p_{n,k}(t)\,|\varepsilon(t,x)|\;|t-x|^{r+2}\,dt\leqslant$$

$$\leqslant(n-1)\,K(x)\sum_{\substack{2i+j\leq r\\ i,\,j\geq 0}}(n+1)^i\left(\sum_{k=0}^{\infty} b_{n,k}(x)\,|[k-(n+1)\,x]|^{2j}\right)^{1/2}$$

$$\cdot\left(\sum_{k=0}^{\infty} b_{n,k}(x)\left(\int_0^{\infty} p_{n,k}(t)\,|\varepsilon(t,x)|\;|t-x|^{r+2}\,dt\right)^2\right)^{1/2},$$

where $K(x)=\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}} |q_{i,j,r}(x)|$.

For a given $\varepsilon>0$ $\exists$ a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. For $|t-x|\geqslant\delta$, we have $|\varepsilon(t,x)|\leqslant M|t-x|^{\gamma}$.

$$\left(\int_0^\infty p_{n,k}(t)\,|\varepsilon(t,x)|\,|t-x|^{r+2}\,dt\right)^2\leqslant$$

$$\leqslant\left(\int_0^\infty p_{n,k}(t)\,dt\right)\left(\int_0^\infty p_{n,k}(t)\,(\varepsilon(t,x))^2\,(t-x)^{2r+4}\,dt\right)=$$

$$=\frac{1}{n-1}\left(\int\limits_{|t-x|<\delta}+\int\limits_{|t-x|\geqq\delta}\right)p_{n,k}(t)\,(\varepsilon(t,x))^2\,(t-x)^{2r+4}\,dt\leqslant$$

$$\leqslant\frac{1}{n-1}\left(\int\limits_{|t-x|<\delta}p_{n,k}(t)\,\varepsilon^2\,(t-x)^{2r+4}\,dt+\right.$$

$$\left.+\int\limits_{|t-x|\geqq\delta}p_{n,k}(t)\,M^2\,(t-x)^{2r+2\gamma+4}\,dt\right).$$

Therefore, by Lemma 2.3, we have

$$\sum_{k=0}^\infty b_{n,k}(x)\left(\int_0^\infty p_{n,k}(t)\,|\varepsilon(t,x)|\,|t-x|^{r+2}\,dt\right)^2\leqslant$$

$$\leqslant\frac{1}{n-1}\sum_{k=0}^\infty b_{n,k}(x)\int_0^\infty p_{n,k}(t)\,\varepsilon^2\,(t-x)^{2r+4}\,dt+$$

$$+\frac{M^2}{n-1}\sum_{k=0}^\infty b_{n,k}(x)\int\limits_{|t-x|\geqq\delta}p_{n,k}(t)\,(t-x)^{2r+2\gamma+4}\,dt=$$

$$=\varepsilon^2\,O(n^{-(r+3)})+\frac{M^2}{n-1}\sum_{k=0}^\infty b_{n,k}(x)\int\limits_{|t-x|\geqq\delta}p_{n,k}(t)\,\frac{(t-x)^{2c}}{\delta^{2c-(2r+2\gamma+4)}}\,dt\leqslant$$

$$\leqslant\varepsilon^2\,O(n^{-(r+3)})+\frac{M^2}{n-1}\,\frac{1}{\delta^{2c-(2r+2\gamma+4)}}\sum_{k=0}^\infty b_{n,k}(x)$$

$$\cdot\int_0^\infty p_{n,k}(t)\,(t-x)^{2c}\,dt=$$

$$=\varepsilon^2\,O(n^{-(r+3)})+O(n^{-(c+1)}).$$

Hence, by using Lemma 2.1, we get

$$|I_n| \leqslant (n-1)\, K(x) \sum_{\substack{2i+j\leq r \\ i,\, j\geq 0}} (n+i)^{i+j} [n \,.\, O(n^{-j})]^{1/2}$$

$$\cdot \{\varepsilon^2\, O(n^{-(r+3)}) + O(n^{-(c+1)})\}^{1/2} =$$

$$= [\varepsilon^2\, O(1) + O(n^{(r+3-c-1)})]^{1/2} \leqslant \varepsilon\, O(i),$$

choosing $c > r+2$ and n sufficiently large, we get $I_n \to 0$ as $n \to \infty$.

Theorem 3.2. Let $f \in \mathscr{L}$ be bounded on every finite subinterval of $[0, \infty)$ and $f(t) = O(t^{\alpha})$ as $t \to \infty$ for some $\alpha > 0$. If $f^{(r+1)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0, \infty)$, $\eta > 0$ and $(a-\eta, b+\eta)$ is an open interval containing the closed interval $[a, b]$, then for sufficiently large n,

$$\| B_n^{(r)}(f, \cdot) - f^{(r)} \| \leqslant K_1 (\| f^{(r)} \| + \| f^{(r+1)} \|)\, n^{-1} +$$

$$+ K_2\, n^{-1/2}\, \omega_{f^{(r+1)}}(n^{-1/2}) + O(n^{-s}) \text{ for any } s > 0$$

where the constants K_1 and K_2 are independent of f and n, $\omega_f(\delta)$ is the modulus of continuity on $(a-\eta, b+\eta)$ and $\| \cdot \|$ is the sup-norm on $[a, b]$.

Proof. For Taylor's finite expansion of f, we have

$$f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i\,!} (t-x)^i + \frac{\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\}}{(r+1)\,!} (t-x)^{r+1} \chi(t) +$$

$$+ g(t, x)\, (1 - \chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta, b+\eta)$. For $t \in (a-\eta, b+\eta)$ and $x \in [a, b]$, we have

$$f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i\,!} (t-x)^i + \frac{\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\}}{(r+1)\,!} (t-x)^{r+1}.$$

For $t \in [0, \infty) \setminus (a-\eta, b+\eta)$ and $x \in [a, b]$, we define

$$g(t, x) = f(t) - \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i\,!} (t-x)^i.$$

Now making use of Lemma 2.4, we obtain

$$B_n^{(r)}(f,x)-f^{(r)}(x)=\left[\frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}\sum_{i=0}^{r+1}\frac{f^{(i)}(x)}{i!}\sum_{k=0}^{\infty}b_{n+r,k}(x)\right.$$

$$\left.\cdot\int_0^\infty p_{n-r,k+r}(t)\frac{d^r}{dx^r}(t-x)^i\,dt-f^{(r)}(x)\right]+\left[\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}^{(r)}(x)\right.$$

$$\left.\cdot\int_0^\infty p_{n,k}(t)\frac{\{f^{(r+1)}(\xi)-f^{(r+1)}(x)\}}{(r+1)!}(t-x)^{r+1}\chi(t)\,dt\right]+$$

$$+\left[\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}^{(r)}(x)\int_0^\infty p_{n,k}(t)\,g(t,x)\,(1-\chi(t))\,dt\right]=$$

$$=I_1+I_2+I_3,\ \text{say.}$$

Using Lemma 2.3, we have

$$I_1=\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}\,[f^{(r)}(x)-f^{(r+1)}(x)\,T_{r,n,1}(x)]-f^{(r)}(x)=$$

$$=f^{(r)}(x)\left[\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}-1\right]-$$

$$-f^{(r+1)}(x)\,\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}\,\frac{x(2r+3)+r+1}{n-r-2}.$$

Hence,

$$\|I_1\|\leqslant K_1(\|f^{(r)}\|+\|f^{(r+1)}\|)\,n^{-1}.$$

Next, using Lemma 2.2, we get

$$|I_2|\leqslant\frac{n-1}{n}\sum_{k=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^i\,|[K-(n+1)x]|^j\,\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\,b_{n,k}(x)$$

$$\cdot\int_0^\infty p_{n,k}(t)\,\frac{|f^{(r+1)}(\xi)-f^{(r+1)}(x)|}{(r+1)!}\,|t-x|^{r+1}\chi(t)\,dt\leqslant$$

$$\leqslant\frac{M_1(n-1)}{n(r+1)!}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^i\sum_{k=0}^{\infty}b_{n,k}(x)\,|[k-(n+1)x]|^j$$

$$\cdot\int_0^\infty p_{n,k}(t)\left(1+\frac{|t-x|}{\delta}\right)\omega_{f^{(r+1)}}(\delta)\,|t-x|^{r+1}\,dt\quad\forall\ \delta>0=$$

$$= \frac{M_1 (n-1)}{n (r+1)\,!} \sum_{\substack{2i+j \leq r \\ i,\, j \geq 0}} (n+1)^i \sum_{k=0}^{\infty} b_{n,k}(x)\, |\, [k-(n+1)\, x]\, |^j\, \omega_{f^{(r+1)}}(\delta)$$

$$\cdot \int_0^{\infty} p_{n,k}(t) \left\{ |\, t-x\,|^{r+1} + \frac{|\, t-x\,|^{r+2}}{\delta} \right\} dt$$

where

$$M_1 = \sup_{\substack{2i+j \leq r \\ i,\, j \geq 0}} \; \sup_{x \in [a,\, b]} \frac{|\, q_{i,j,r}(x)}{x^r (1+x)^r} .$$

Now, we shall show that for $s=0, 1, 2, \ldots$

$$\frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x)\, |\, [k-(n+1)\, x]\, |^j \int_0^{\infty} p_{n,k}(t)\, |\, t-x\,|^s\, dt = O\,(n^{(j-s)/2}).$$

We prove it as follows:

$$\frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x)\, |\, [k-(n+1)\, x]\, |^j \int_0^{\infty} p_{n,k}(t)\, |\, t-k\,|^s\, dt \leqslant$$

$$\leqslant \frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x)\, |\, [k-(n+1)\, x]\, |^j$$

$$\cdot \left[\left(\int_0^{\infty} p_{n,k}(t)\, dt \right)^{1/2} \cdot \left(\int_0^{\infty} p_{n,k}(t)\, (t-x)^{2s}\, dt \right)^{1/2} \right] =$$

$$= \frac{\sqrt{n-1}}{n} \sum_{k=0}^{\infty} b_{n,k}(x)\, |\, [k-(n+1)\, x]\, |^j \left(\int_0^{\infty} p_{n,k}(t)\, (t-x)^{2s}\, dt \right)^{1/2} \leqslant$$

$$\leqslant \sqrt{\frac{n-1}{n}} \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x)\, [K-(n+1)\, x]^{2j} \right)^{1/2}$$

$$\cdot \left(\sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} p_{n,k}(t)\, (t-x)^{2s}\, dt \right)^{1/2} =$$

$$= (n+1)^j \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \left[\frac{k}{n+1} - x \right]^{2j} \right)^{1/2}$$

$$\cdot \left(\frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} p_{n,k}(t)\,(t-x)^{2s}\,dt \right)^{1/2} =$$

$$= O(n^{j/2}) \cdot O(n^{-s/2}) = O(n^{(j-s)/2}),$$

uniformly in x, by Lemma 2.1 and Lemma 2.3. Hence

$$\| I_2 \| \leqslant M_1 \frac{\omega_{f^{(r+1)}}(\delta)}{(r+1)\,!} \sum_{\substack{2i+j\leq \\ i,\,j\geq 0}} (n+1)^i \left\{ O(n^{(j-r-1)/2}) + \frac{1}{\delta} O(n^{(j-r-2)/2}) \right\}.$$

Choosing $\delta = n^{-1/2}$, we get

$$\| I_2 \| \leqslant K_2\, n^{-1/2}\, \omega_{f^{(r+1)}}(n^{-1/2}).$$

Since $t \in [0, \infty) \setminus (a-\eta, b+\eta)$, we can choose a $\delta > 0$ such that $|t-x| \geqslant \delta$ for $x \in [a, b]$. Using Lemma 2.2, we obtain

$$| I_3 | \leqslant \frac{n-1}{n} \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r \\ i,\,j\geq 0}} (n+1)^i \,|\, [k-(n+1)\,x] \,|^j \frac{| q_{i,j,r}(x) |}{x^r (1+x)^r}\, b_{n,k}(x)$$

$$\cdot \int_{|t-x|\geq\delta} p_{n,k}(t)\,|\,g(t,x)\,|\,dt.$$

If β is any integer $\geqslant \max\{\alpha, (r+1)\}$, we can find a constant M_2 in such a way that $|\,g(t,x)\,| \leqslant M_2\,|\,t-x\,|^{\beta}$, for $|\,t-x\,| \geqslant \delta$. Using Cauchy's inequality, Lemma 2.1 and Lemma 2.3, we get

$$| I_3 | \leqslant M_1 \frac{n-1}{n} \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r \\ i,\,j\geq 0}} (n+1)^i \,|\, [k-(n+1)\,x] \,|^j\, b_{n,k}(x)$$

$$\cdot \int_{|t-x|\geq\delta} p_{n,k}(t)\, M_2\,|\,t-x\,|^{\beta}\,dt \leqslant$$

$$\leqslant M_3 \frac{n-1}{n} \sum_{\substack{2i+j\leq r \\ i,\,j\geq 0}} (n+\mathrm{i})^i \sum_{k=0}^{\infty} |\, [k-(n+\mathrm{i})\,x] \,|^j\, b_{n,k}(x)$$

$$\cdot \int_{|t-x|\geq\delta} p_{n,k}(t)\, \frac{(t-x)^{2m}}{\delta^{2m-\beta}}\,dt =$$

$$=\sum_{\substack{2i+j\leq r\\ i,\, j\geq 0}} (n+1)^i\, O(n^{j/2})\, O(n^{-m}) = O(n^{(r-2m)/2}),$$

uniformly on $[a, b]$, where m is a natural number bigger than $\beta/2$. Combining the estimates of I_1, I_2 and I_3, we get the required result.

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