ABOUT A CLASS OF FINITE GROUPS

Ion ARMEANU
University of Bucharest, Physics Faculty, Mathematics Dept., Bucharest-Magurele,
P.O. Box MG-11, ROMANIA

Summary : This note is concerned to prove some interesting facts about the groups G who have the property that $N_G(ax)$ is subnormal in $N_G(a)$ for every $a,x \in G$ such that $ax = xa$, where a has odd order and the order of x is a power of 2.

BİR SONLU GRUP SINIFI HAKKINDA

Özet : Bu çalışmada şu özeliği taşıyan sonlu G gruplarına ilişkin bazı ilginç sonuçlar elde edilmektedir: "$a,x \in G$, a nın mertebesi tek, x in mertebesi 2 nin kuvveti biçiminde ve $ax = xa$ olmak üzere, her a, x çifti için $N_G(ax)$, $N_G(a)$ nın bir normal alt grubudur".

In this note we will use only finite groups and the notations and definitions will be those of [3].

Definitions :

a) We will say that a group G is an A-group if for every $a \in G$ of odd order and for every $x \in G$ of order a power of 2, such that $ax = xa$, then $N_G(ax)$ is subnormal in $N_G(a)$.

b) We will say that a group is a **Q**-group if all its irreducible characters are rational valued.

c) We will say that a group is a **Q**A-group if it is a **Q**-group and an A-group too.

Proposition 1. Let G be an A-group. Let $a, x \in G$ be as in the Definitions. Then:

a) $N_G(ax) \leqslant N_G(a)$.

b) $C_G(ax)$ is subnormal in $N_G(a)$.

c) Let H be a 2-Sylow group of $N_G(a)$ and H_0 be the 2-Sylow group of $C_G(a)$ such that $H_0 \leqslant H$. Then $H \cap N_G(ax) = N_2$ is a 2-Sylow group of $N_G(ax)$ and $H_0 \cap C_G(ax) = C_2$ is a 2-Sylow group of $C_G(ax)$.

The proof is obvious.

For the next we will use the already introduced notations.

Theorem 2. Let G be a $\mathbf{Q}A$-group. Then $H_0 \leqslant H$ is fusion free.

Proof. Let be $x \in H_0$ and $b \in H \setminus H_0$. We will show that there exists a $h \in H_0$ such that $b^{-1}xb = h^{-1}xh$. Let $u \in \mathrm{Aut}_2(a)$ be the nontrivial inner automorphism given by b, where $\mathrm{Aut}_2(a)$ is the 2-Sylow group of $\mathrm{Aut}(a)$. Since G has rational valued characters we have that

$N_G(ax)/C_G(ax) \approx \mathrm{Aut}(ax) \approx \mathrm{Aut}(a) \times \mathrm{Aut}(x)$ (see [3], pg. 11) and that

$N_2/C_2 \approx N_2 \cdot C(ax)/C(ax)$ (see [4], pg. 56).

Therefore there exists a $b_u \in N_2$ such that its image in $\mathrm{Aut}_2(a) \times \mathrm{Aut}_2(x)$ to be $u.1$, consequently $b_u^{-1}\, ab_u = b^{-1}\, ab$ and b_u commutes with x.

Since b and b_u lead to the same automorphism of $\langle a \rangle$, there exists a $h \in H_0$ such that $b = b_u h$. Then

$$b^{-1}xb = h^{-1}b_u^{-1}xb_u\, h = h^{-1}xh\,.$$

Theorem 3. Let G be a $\mathbf{Q}A$-group. Then H_0 is a $\mathbf{Q}$-group.

Proof. Let $f_a : N_G(a) \to \mathrm{Aut}(a)$ be given by $f_a(x)(a) = x\,ax^{-1}$. Since G has rational valued characters, f_a is an epimorphism. Let $h \in H_0$, and let z, w be the generators of $\mathrm{Aut}(h)$. For

$$f_{ah} : N_G(ah) \to \mathrm{Aut}(ah) \approx \mathrm{Aut}(a) \times \mathrm{Aut}(h)$$

there exist $x, y \in N_G(ah)$ such that $f_{ah}(x) = z$ and $f_{ah}(y) = w$. Since $\mathrm{Aut}(h)$ is a 2-group it follows that any odd powers of z and w are generators for $\mathrm{Aut}(h)$ too. Therefore if $|x| = 2^j q$ and $|y| = 2^k r$, with q and r odd integers, considering $x_1 = x^q$ and $y_1 = y^r$ it follows hat $f_{ah}(x_1)$ and $f_{ah}(y_1)$ are generators for $\mathrm{Aut}(h)$, besides h, $x_1, y_1 \in C(a) \cap N(ah)$. Since G is an A-group, using the Sylow's theorem we obtain that there exist u, $v \in C(a) \cap N(ah)$ such that the elements $x_2 = ux_1u^{-1}$ and $y_2 = vy_1v^{-1}$ belong to the 2-Sylow group $H_0 \cap N(ah)$ of $N(ah)$. Besides $f_{ah}(x_2) = f_{ah}(x_1)$ and $f_{ah}(y_2) = f_{ah}(y_1)$. Therefore $f_{ah}(x_2)$ and $f_{ah}(y_2)$ generate $\mathrm{Aut}(h)$.

Remark. In particular, for $a = 1$ we obtain that for a $\mathbf{Q}A$-group it holds the old standing conjecture (see [3], pg. 13) that asserts that for a $\mathbf{Q}$-group the 2-Sylow subgroups are $\mathbf{Q}$-groups too. In fact, at this moment I do not know an example of a $\mathbf{Q}$-group which is not an A-group.

Proposition 4. a) Let G be an A-group and $H \leqslant G$. Then H is also an A-group.

b) Let G be a $\mathbf{Q}A$-group and $H \leqslant G$ fusion free. Then H is also a $\mathbf{Q}A$-group.

Proof. We know that $N_G(ax)$ is subnormal in $N_G(a)$. Then $N_H(ax) = N_G(ax) \cap H$ is subnormal in $N_H(a) = N_G(a) \cap H$ (see [4], pg. 127).

Proposition 5. Let G be a $\mathbf{Q}A$-group with abelian 2-Sylow group. Then:

a) Any 2-Sylow group is isomorphic with $\mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ and the Schur index $m_{\mathbf{Q}}(X) = 1$, $\forall\ X \in \operatorname{Irr}(G)$.

b) G is strong real.

Proof. Since the 2-Sylow groups are abelian $\mathbf{Q}$-groups it follows immediately that they are isomorphic with $\mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$. Through the Brauer-Speiser Theorem (see [5], pg. 9) and Fein-Yamada Theorem (see [5], pg. 143) we obtain that $m_{\mathbf{Q}}(X) = 1$ for every $X \in \operatorname{Irr}(G)$. We get b through the Theorem 2.4 of [2].

Remark. The 2-$\mathbf{Q}A$-groups are exactly the 2-$\mathbf{Q}$-groups.

Proposition 6. a) $\mathbf{Z}_2$ wr ... wr $\mathbf{Z}_2$ is a $\mathbf{Q}A$-group.

b) A 2-group is a $\mathbf{Q}A$-group if and only if it can be embedded without fusion in a direct product of $\mathbf{Z}_2$ wr ... wr $\mathbf{Z}_2$ (wr means wreath product).

For the proof see [1].

Proposition 7. Let G be a $\mathbf{Q}A$-group with nonabelian dihedral resp. quaternionic 2-Sylow groups. Then the 2-Sylow groups are isomorphic with D_8 resp. Q_8.

Proof. D_8 resp. Q_8 are the only dihedral resp. quaternionic nonabelian groups whose characters are rational valued.

REFERENCES

[1] ALEXANDRU, V. and ARMEANU, I. : *Sur les caractères d'un groupe fini*, C.R. Acad. Sci. Paris, **298** (1984), Serie I, No. 6.

[2] GOW, R. : *Real-Valued and 2-Rational Group Characters*, J. of Algebra, **61** (2) (1979), 388-413.

[3] KLETZING, D. : Structure and Representations of Q-groups, Lecture Notes in Mathematics, Springer-Verlag, 1984.

[4] ROSE, J.S. : A Course in Group Theory, Cambridge, 1978.

[5] YAMADA, T. : The Schur Subgroup of the Brauer Group, Lecture Notes in Mathematics 397, Springer-Verlag, 1974.