İstanbul Üniv. Fen Fak. Mat. Der., **54** (1995), 35-40

ON THE ROTATION ENTROPY FUNCTION FOR FLOWS

Bünyamin AYDIN
Department of Mathematics, Yüzüncü Yıl University, Van-TURKEY

Summary : In this paper, we recall some fundemantel properties of rotations sets for flows without going into details and all proofs. After that we define the rotation entropy function for flows. Finally we prove some important prorperties of this function.

AKIŞLARA GÖRE ROTASYON ENTROPİ FONKSİYONU HAKKINDA

Özet : Bu çalışmada, akışlara göre rotasyon kümelerinin özelikleri ayrıntıya girilmeksizin verilmekte, daha sonra akışlara göre rotasyon entropi fonksiyonu tanımlanarak buna ilişkin bazı önemli özeliklerin ispatı verilmektedir.

1. Flows

1. Definition. Let $(X, \mathbf{A}, \mu, \{\Phi_t\}_{t\in \mathbf{n}})$ be a dynamical system with continuous time where $(X, \mathbf{A}, \mu)$ is a measure space and $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$ is a group of transformations defined on the measure space $(X, \mathbf{A}, \mu)$. If for every $t \in \mathbf{R}$ the transformations $\{\Phi_t\}_{t\in\mathbf{R}}$ satisfy the properties

(i) For all $t_1, t_2 \in \mathbf{R}$ $\Phi_{t_1+t_2} = \Phi_{t_1} + \Phi_{t_2}$;

(ii) For every measurable function $\Psi(x)$ defined on the measure space $(X, \mathbf{A}, m)$ the function $(\Phi_t . x)$ defined on the cartesian product $X \times \mathbf{R}$ is measurable, then the set $\{\Phi_t\}_{t\in\mathbf{R}}$ is called a flow.

2. Definition. If $g(\Phi_t, x) = g(x)$ for $x \in X$ and $t \in \mathbf{R}$ then the measurable function g is said to be invariant with respect to flow $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$.

3. Theorem. If $\Phi = \{\Phi_t : X \to X\}$ is a measure preasuring transformation of the probability measure space $(X, \mathbf{A}, \mu)$ then the following statements are equivalent:

(i) Φ is ergodic.

(ii) For every $A, B \in \mathbf{A}$ with $m(A) > 0$, $m(B) > 0$ there exist $n > 0$ and $t \in \mathbf{R}$ with $m(\Phi^{nt} B) > 0$ [6].

4. Definition. Let (X, A, m) be topological measure space. A measure-preserving transformation Φ of (X, A, m) is called ergodic with respect to $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$ flows if the only members B of A with $\Phi^{-1} B=B$ satisfy $m(B)=0$ or $m(B)=1$.

5. Theorem. If Φ-flow is ergodic then its factor is ergodic [1].

2. Rotational Sets With Respect To Flows

We shall denote the class of liftings of all continuous flows by $\mathbf{C}_m$.

6. Definition. Let $\square\, \mathrm{p}_n(\Phi)$ denote the rotational number set with respect to $\Phi = \{\Phi_t\}_{t\in\mathbf{R}}$ flow of accumulation points of the $\mathbf{R}^n$

$$\left\{\frac{\Phi^s(x) - x}{s} : x \in \mathbf{R}^n,\ s \in (0, t]\right\}.$$

7. Definition. The rotation set $\mathrm{p}(\Phi)$ with respect to flows is defined by $v \in \mathrm{p}(\Phi)$ if and only if there are sequences x_i, $y_i \in \mathbf{R}^n$ and $s \in (0, t]$ such that

$$\lim_{s\to\infty} \frac{\Phi^s(x_i) - \Phi^s(y_i)}{s} = v.$$

8. Proposition. If $\Phi \in \mathbf{C}_m$, then $\mathrm{p}_n(\Phi) \subset \mathrm{p}(\Phi)$ [1].

9. Property. The set $\mathrm{p}(\Phi)$ has better properties than the set $\mathrm{p}_n(\Phi)$ [2].

10. Theorem. If $\Phi \in \mathbf{C}_m$ then the rotation set with respect to flows $\rho(\Phi)$ is connected.

Proof. Suppose that $\mathrm{p}(\Phi)$ is not connected. Then there exist disjoint sets $A, B \subset \rho(\Phi)$ such that $\mathrm{p}(\Phi) = A \cup B$ and both A and B are closed in $\mathrm{p}(\Phi)$. A and B are also compact. There exist open neighbourhoods U and V of A and B respectively, such that U and V are compact and disjoint sets and $\delta = \mathrm{dist}(U, V) > 0$. Therefore $v \in A$, $w \in B$. By the definition of $\mathrm{p}(\Phi)$, there exist sequences of integers $s_i \to \infty$, $k_j \to \infty$ and sequences (x_i), (y_i) of points of $\mathbf{R}^+$, such that

$$\mathrm{hm}_{i\to\infty} \frac{\Phi^{s_i}(y_i) - x_i}{s_i} = v \quad \text{and} \quad \mathrm{hm}_{i\to\infty} \frac{\Phi^{s_i}(y_i) - y_i}{k_i} = w.$$

There exists M such that $\| T(t) - t \| \leq M$ for all $t \in \mathbf{R}^+$. Therefore if we take $z \in \mathbf{R}^+$ and write $t = T^s(z)$ then

$$\left\| \frac{\Phi^{s+1}(z) - z}{s+1} - \frac{\Phi^{s}(z) - z}{s} \right\| = \left\| \frac{\Phi(t) - z}{s+1} - \frac{t-z}{s} \right\| \leq$$

$$\leq \left\| \frac{\Phi(t) - z}{s+1} - \frac{\Phi(t) - z}{s} \right\| + \left\| \frac{\Phi(t) - z}{n} - \frac{t-z}{n} \right\| \leq$$

$$\leq \| \Phi(t) - z \| \left(\frac{1}{s} - \frac{1}{s+1} \right) + \frac{M}{s} =$$

$$= \frac{1}{s} \left(\left\| \frac{\Phi^{s+1}(z) - z}{s+1} \right\| M \right).$$

Since

$$\| \Phi^{s+1}(z) / z \| \leq \| \Phi^{s+1}(z) - \Phi^{s}(z) \| + \ldots + \| \Phi(z) - z \| \leq (s+1) M$$

we get

$$\left\| \frac{\Phi^{s+1}(z) - z}{s+1} - \frac{\Phi^{n}(z) - z}{n} \right\| \leq \frac{2M}{s}.$$

Since $\rho(\Phi)$ is contained in an open set $U \cup V$, there exists M such that

$$\frac{\Phi^{s}(z) - z}{s} \in U \cup V$$

for all $n \in \mathbf{N}$ and $z \in \mathbf{R}^{+}$. Take i such that $2M/s_i \subset \delta_i$, $s_i \geq N$, $(\Phi^{s_i}(x_i) - x_i)/s_i \in U$ and take j such that $k_j \geq s_i$ and

$$\frac{\Phi^{k_j}(y_j) - y_{ij}}{k_i} \in V.$$

Then

$$\left\| \frac{\Phi^{s+1}(x_i)}{s+1} - \frac{\Phi^{s}(x_i) - x_i}{s} \right\| \leq \operatorname{dist}(\bar{u}, \bar{v})$$

for all $t \in \mathbf{R}$. Hence $\rho(\Phi)$ is connected.

11. Definition. A subset E of X is called (t, ε)-rotational spanning set with respect to flow $\Phi = \{\Phi_t\}_{t \in \mathbf{R}}$ if for all $x \in A$ there exists $y \in E$ so that $|x - y| < \varepsilon$ and

$$\left| \frac{\Phi^{s}(x) - \Phi^{s}(y)}{s} \right| < \varepsilon, \quad 0 < s \leq t.$$

12. Proposition. There exists a minimal (t, ε)-rotational spanning set with respect to flows and finite.

Proof. For fixed s and ε, given $x \in A$ there exists $0 < \delta < \varepsilon$ so that

$$\left|\frac{\Phi^s(x) - \Phi^s(y)}{s}\right| < \varepsilon, \quad 0 < s \leq t$$

and $y \in \beta(x, \delta)$. Since A is compact, there are k balls with centers $x_1, x_2, \ldots, x_k$ that cover the whole annulus, the set $\{x_1, x_2, \ldots, x_k\}$ is (t, ε)-rotational spanning with respect to flows.

13. Definition. A subset G of X is called (t, ε)-rotational separated sets with respect to flows if for any distinct $x, y \in G$ there is an s such that $|x - y| > \varepsilon$ and

$$\left|\frac{\Phi^s(x) - \Phi^s(y)}{s}\right| > \varepsilon, \quad 0 < s \leq t.$$

14. Proposition. Let $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $0 < t_1 \leq t$, $0 < t_2 \leq t$. If

(i) $t_1 \leq t_2$ then $r_{t_1, \varepsilon}(E) \leq r_{t_2, \varepsilon}(E)$,

(ii) $t_1 \leq t_2$ then $S_{t_1, \varepsilon}(G) \leq S_{t_2, \varepsilon}(G)$,

(iii) $\varepsilon_1 < \varepsilon_2$ then $r_{\varepsilon_2, t_1}(E) \leq r_{\varepsilon_1, t_2}(G)$ [5].

15. Definition. We denote by $r_s(\varepsilon, E)$ the minimal conditional of a set which (s, ε)-spanns E and write

$$r(\varepsilon, E) = \limsup_{t \to \infty} \frac{1}{t} \log r_t(\varepsilon, E).$$

16. Proposition. If $E \subset X$ is a rotational spanning set with respect to flows theu

$$h(\Phi) = \lim_{\varepsilon \to 0} r(\varepsilon, E) = \lim_{\varepsilon \to 0} s(\varepsilon, G) = h^2(\Phi).$$

Proof. This is a consequence of $r_t(\varepsilon, E) \leq s_t(\varepsilon, E) \leq r_s(\varepsilon/2, E)$. The left inequality is obvious; any maximal (t, ε)-separated rotational set with respect to flows $K \subset A$ $(s, E/2)$-spanning rotational set with respect to flows for E. For each $x \in E$ there is a $k(x) \in K$ such that

$$\left|\frac{\Phi^s(x) - \Phi^s(y)}{s}\right| < \frac{\varepsilon}{2} \quad \text{for } 0 < s \leq t.$$

If x_1, x_2 are two distinct points of a (t, ε)-separated rotational set with respect to flows then $k(x_1) \neq k(x_2)$ since otherwise

$$\left|\frac{\Phi^s(x_1) - \Phi^s(x_2)}{s}\right| \leq \varepsilon \quad \text{for } 0 < s \leq t.$$

Hence Card $E \leq$ Card K and therefore $s_t(\varepsilon, E) \leq r_t(\varepsilon/2, K)$.

17. Proposition. There exist maximal (t, ε)-rotational separated sets with respect to flows and finite [3].

18. Definition. For a subset G of X one denotes by $S_t(\varepsilon, G)$ the largest cardinality of a (t, ε)-separated subset of G. Write

$$S(\varepsilon, G) = \lim_{t \to \infty} \frac{1}{t} \log S_t(\varepsilon, G).$$

19. Proposition. If $G \subset X$ is rotational set with respect to flows then

$$h^2(\Phi) = \lim_{\varepsilon \to \infty} S_t(\varepsilon, G) \ [3].$$

20. Theorem.

$$\lim_{\varepsilon \to 0} h^1_\varepsilon(\Phi) = h^1(\Phi),$$

$$\lim_{\varepsilon \to 0} h^2_\varepsilon(\Phi) = h^2(\Phi),$$

$$h^1(\Phi) = h^2(\Phi) = h(\Phi) \ [3].$$

21. Proposition. $h(\Phi^t) = th(\Phi)$ for any $t \in \mathbf{R}^+$.

Proof. Clearly $r_{t_1, \varepsilon}(E) \leq r_{t_2, \varepsilon}(E)$, so it follows that $h(\Phi)^t \leq h(\Phi)$. Given $\varepsilon > 0$ choose $\delta > 0$ such that $|x - y| < \varepsilon$ and

$$\left| \frac{\Phi^s(x) - \Phi^s(y)}{s} \right| > \varepsilon, \quad 0 < s \leq t.$$

One sees then on (t, ε)-spanning set for E with respect to flows $\Phi = \{C\}_{t \in \mathbf{R}}$. Hence $r_{t_1, \varepsilon}(E) \leq r_{t_2, \varepsilon}(E)$ and we get $h(\Phi)^t \leq h(\Phi)$.

22. Proposition. Let φ and γ be two flows. Then $h(\varphi \times \gamma) = h(\varphi) + h(\gamma)$.

Proof. If k_i $(i = 1, 2)$ is a (t, ε)-spanning set with respect to flows for E_i then $k_1 \times k_2$ is a (t, ε)-spanning set for flows for $E_1 \times E_2$ with respect to $\varphi \times \gamma$.

$$r_s(\varepsilon, E_1 \times E_2, \varphi \times \gamma) = r_s(\varepsilon, E_1, \varphi) + r_s(\varepsilon, E_2, \gamma),$$

$$h(\varphi \times \gamma, E_1 \times E_2) \leq h(\varphi, E_1) + h(\gamma, E_2) \quad \text{(I)}.$$

If $F_i \subset E_i$ is an (s, ε)-separated set with respect to flows $\Phi = \{\varphi_t\}_{t \in \mathbf{R}}$ then $F_1 \times F_2 \subset E_1 \times E_2$ is (s, ε)-separated with respect to $E_1 \times E_2$. Hence

$$S_s(\varepsilon, E_1 \times E_2, \varphi \times \gamma) \geq S_s(\varepsilon, E_1, \varphi) + S_s(\varepsilon, E_2, \gamma)$$

and

$$h(\varphi \times \gamma, E_1 \times E_2) \geq h(\varphi, E_1) + h(\gamma, E_2) \quad \text{(II)}$$

and we get $h(\varphi \times \gamma) = h(\varphi) + h(\gamma)$ from (I) and (II).

REFERENCES

[1] AYDIN, B. and GÜLCÜ, A.: *Akışlara Göre Rotasyon Kümeleri*, Yüzüncü Yıl Üniversitesi Fen-Edebiyat Fakültesi Fen Bilimleri Dergisi, **3** (1992), Sayı: 3, 1018-3710.

[2] BOTELHO, F. : *Rotational Entropy For Annulus Endomorphism*, Pasific J. Math., **151** (1991), No: 1, 1-19.

[3] BOWEN, R. : *Entropy For Group Endomorphism and Homogeneous Spaces*, Trans. Amer. Math. Soc., **153** (1971).

[4] FRANKS, J. and MISIUREWICH, M. : *Rotation Sets of Toral Flows*, Proc. Amer. Math. Soc., 109 (1982), 243.

[5] MISIUREWICH, M. and ZIEMAN, K. : *Rotation Sets and Ergodic Measures For Torus Homeomorphism*, Fundamenta Math., 137 (1991).

[6] WALTERS, P. : Introductim to Ergodiç Theory, Graduate Texts in Math., Vol. 79, Springer-Verlag, Berlin, 1982.