RATIONAL ALTERNATING GROUPS

Ion ARMEANU
University of Bucharest, Physics Faculty, Mathematics Dept., Bucharest-Magurele,
P.O. Box MG-11, ROMANIA

Summary : In this note we shall prove that only A_{10} is a rational alternating group.

RASYONEL ALTERNE GRUPLAR

Özet : Bu çalışmada tek rasyonel alterne grubun A_{10} olduğu ispat edilmektedir.

The notations and definitions will be those of [1] and [2].

Definition (see [2]). A rational group (or a **Q**-group) is a finite group all whose irreducible characters are rational valued.

Proposition 1 (see [2]). Let G be a finite group.

(i) Then G is rational if and only if for every $x \in G$, $N(\langle x \rangle) / C(x) \simeq \simeq \operatorname{Aut}(\langle x \rangle)$.

(ii) An element $x \in G$ is rational iff $N(\langle x \rangle) / C(x) \simeq \operatorname{Aut}(\langle x \rangle)$.

Theorem 2. A_{10} is the only nontrivial rational alternating group.

Proof. It is well known that the symmetric groups S_n are rational groups. Since the alternating group A_n is normal in S_n, it contains full conjugacy classes of S_n. Hence, for an elements x of A_n to be rational it is necessary that the conjugacy class of x, $\mathrm{cl}(x)$, in S_n splits in A_n. The length of $\mathrm{cl}(x)$ is the index of the centralizer of x in the corresponding group. Clearly $C_{A_n}(x) = C_{S_n}(x) \cap A_n$. Since $|S_n : A_n| = 2$, either $C_{A_n}(x) = C_{S_n}(x)$ or $|C_{S_n}(x) : C_{A_n}(x)| = 2$. Therefore the S_n conjugacy class of x splits in A_n iff $C_{A_n}(x) = C_{S_n}(x)$.

We prove now that $C_{A_n}(x) = C_{S_n}(x)$ if and only if the type of x (see [1]), $T(x) = (t_1, \dots, t_s)$ has t_i odd and pairwise different.

Suppose $C_{A_n}(x) = C_{S_n}(x)$. Since x commutes with its cyclic factors then x cannot have cyclic factors of even length. If x has two cyclic factors $(i_1, \dots, i_k)$

and $(j_1, \ldots, j_k)$ of the same length, then $(i_1\, j_1) \ldots (i_k\, j_k)$ belongs to $C_{S_n}(x)$ hut is not in A_n.

Suppose now that $T(x)=(t_1, \ldots, t_s)$ with t_i odd and pairwise different. Then $|C_{S_n}(x)| = \Pi\ t_i$ is odd, hence $C_{S_n}(x) = C_{A_n}(x)$.

Let $x = (1\,2 \ldots k) \ldots (j \ldots n)$ product of odd lengths disjoint cycles. Then $y = (2\,k)\,(3\,(k-1)) \ldots (j+1\; n) \ldots$ inverts x. Clearly, y is not in A_n if and only if the number of cyclic factors of x having length congruent to 3 mod 4 is odd. Suppose $y \notin A_n$ and there is a $z \in A_n$ which inverts x. Then $(z^{-1}\,y)\,x\,(z^{-1}\,y)^{-1} = x$ and $z^{-1}\,y \notin A_n$, hence $C_{S_n}(x) \neq C_{A_n}(x)$, contradiction. We shall study how the existence of such conjugacy classes.

For $n = 4l + 1$, $l \geq 2$ we have the partition $(4l - 3, 3, 1)$. For $n = 4l + 2$, $l \geq 4$ the partition $(4\,(l-1) - 3, 5, 3, 1)$. For $n = 4l + 3$ the partition $(4l + 3)$. For $n = 4l$ the partitions $(3, 1)$ resp. $(4l - 3, 3)$. Hence no alternating group A_n with n like before can be rational.

Therefore it remains to study the rationality of A_5, A_6, A_{10}, A_{14}.

For A_5 the partition (5) has odd and different elements. Let $x = (12345)$. Since $\mathrm{Aut}(\langle x \rangle) \simeq \mathbf{Z}_4$, A_5 is not rational by Prop. 1.

For A_6 we have the partition (5, 1) and as before A_6 is not rational.

For A_{14} we have the partition (13, 1) and as before the element $(1 \ldots 13)$ is not rational.

For A_{10} the only partitions with odd and different elements are (9, 1) and (7, 3). Since the elements of these types are rational, it follows that A_{10} is rational.

Remark. At this moment we do not know examples of rational simple groups other than $\mathbf{Z}_2$ and A_{10}. We believe that only these groups are rational simple groups.

REFERENCES

[1] HUPPERT, B. : Endliche Gruppen, Springer-Verlag, 1967.

[2] KLETZING, D. : Structure and Representations of Q-Groups, Lecture Notes in Mathematics, Springer-Verlag, 1984.