ON SOME MULTIVALUED SYSTEMS AND FUNDAMENTAL RELATIONS

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Summary : In this paper we introduce the definition of the *n*-hypering relation and prove some results related to it.

ÇOK DEĞERLİ BAZI SİSTEMLER VE TEMEL BAĞINTILAR HAKKINDA

Özet : Bu çalışmada "*n*-hypering relation" tanımı ithal edilmekte ve bununla ilgili bazı sonuçlar ispat edilmektedir.

1. INTRODUCTION

In this paper we study the largest classes of multivalued systems that satisfy group-like and ring-like axioms. The first multivalued system is the hypergroup which is defined as follows: $<H, \cdot>$ is a hypergroup if $\cdot : H \times H \rightarrow \rho(H)$ is a hyperoperation such that the associative law and the reproduction axiom are satisfied, i.e. $x.(y.z)=(x.y).z$ and $x.H=H.x=H$ for all $x, y, z \in H$. The multiplications $x.(y.z)$ and $x.H$ are understood as $\bigcup_{t \in y.z} \{x \cdot t\}$ and $x.H = \bigcup_{h \in H} \{x.h\}$ respectively, and $\rho(H)$ is the power set of H. Special cases of those structures are: canonical hypergroups [9], reversible hypergroups, cogroups, polygroups [1], join spaces [12], [2], feebly canonical hypergroups [3] and others as for example in [4], [5], [6]. The second multivalued system is the hyperring in the general sense: $<R, +, \cdot>$ is a hyperring in the general sense if $+ : R \times R \rightarrow \rho(R)$, $\cdot : R \times R \rightarrow \rho(R)$ are two hyperoperatins such that $<R, +>$ is a hypergroup and $(.)$ is associative hyperoperation which is distributive with respect to $(+)$ not necessarily strong, i.e. $x(y+z) \subset xy + xz$, where $x(y+z)$ is understood as $\bigcup_{t \in y+z} \{x \cdot t\}$ for all $z, x, y \in R$. If only the multiplication $(.)$ is a hyperoperation and the addition $(+)$ is a usual operation then we say that R is a multiplicative hyperring [13], [11]. If only $(+)$ is a hyperoperatin we shall say that R is an additive hyperring; a special case of this is the hyperring introduced by Krasner in [7] and studied by Miltas [10] and others. For more details see [13], where the above definitions have been taken.

2. The n-Fundamental Relation

Definition 2.1. [8] Let $<R, +, \cdot>$ be a hyperring in the general sense. For a given natural number $n \neq 0$ we define in R the n-fundamental relation as follows, which we denote by $\underline{n}$:

(i) $a\underline{n}a$ for all $a \in R$,

(ii) $a\underline{n}b$, $a \neq b$ if we can find elements $x_0, x_1, \dots x_k \in R$ where $x_0 = a$, $x_k = b$, and $i_1, i_2, \dots, i_k > 1$ are natural numbers such that

$$\{x_{s-1}, x_s\} \subset \sum_{\nu=1}^{n^{i_s-1}} \left(\prod_{\mu=1}^{i_s} x_{s\mu\nu}\right), \quad s = 1, \dots, k,$$

where $x_{s\mu\nu} \in R$. It can be easily seen that the n-fundamental relation is an equivalence relation. Let us denote by F_n the quotient set, we call it the n-quotient set, by $F_n(x)$ the n-fundamental class of the element x.

Proposition 2.2. [8] If $\{u, v\} \subset xy$ for some $x, y \in R$, then $u\underline{n}v$.

Proof. From the reproduction axiom with respect to $(+)$ we can choose $y_1, y_2, \dots, y_n \in R$ such that $y \in y_1 + y_2 + \dots + y_n$, then

$$\{u, v\} \subset xy \subset x(y_1 + y_2 + \dots + y_n) \subset xy_1 + xy_2 + \dots + xy_n.$$

Therefore from the definition we obtain $u\underline{n}v$.

Definition 2.3. [13] Let R be a hyperring. In the quotient set F_n we can define two hyperoperations in the usual manner:

$$F_n(x) + F_n(y) = \{F_n(z) : x \in x' + y', \text{ for all } x'\underline{n}x, y'\underline{n}y\}$$

$$F_n(x) \,.\, F_n(y) = \{F_n(z) : z \in x' \,.\, y', \text{ for all } x'\underline{n}x, y'\underline{n}y\}$$

for every $F_n(x)$, $F_n(y)$ of F_n.

Theorem 2.4. [13] $F_n(x) \,.\, F_n(y) = \{F_n(z)\}$, for all $z \in xy$ (see [13]).

Definition 2.5. Let $<R, +, \cdot>$ be a hyperring, then $<R, +, \cdot>$ is called a distributive hyperring if $(+)$ is a distributive operation with respect to $(\cdot)$ as follows:

$$x + (y \cdot z) \subset (x + y) \cdot (x + z) \text{ for all } x, y, z \in R.$$

We are now in a position to prove the following theorem:

Theorem 2.6. [13] Let $<R, +, \cdot>$ be a distributive hyperring, then

$$F_n(x) + F_n(y) = \{F_n(z)\}, \text{ for all } z \in x + y.$$

Proof. Let

$$F_n(x) + F_n(y) = \{F_n(z) : z \in x' + y', \text{ for all } x' \underline{n} x, y' \underline{n} y\}. \tag{1}$$

$x' \underline{n} x \Longrightarrow \exists\ x_0, x_1, \ldots, x_k \in R$ with $x_0 = x', x_k = x$ such that

$$\{x_{s-1}, x_s\} \subset \sum_{\nu=1}^{n^{i_s-1}} \Big(\prod_{\mu=1}^{i_s} x_{s\mu\nu}\Big), \quad s = 1, \ldots, k. \tag{2}$$

$y' \underline{n} y \Longrightarrow \exists\ y_0, y_1, \ldots, y_\lambda \in R$ with $y_0 = y', y_k = y$ such that

$$\{y_{t-1}, y_t\} \subset \sum_{\nu=1}^{n^{i_t-1}} \Big(\prod_{\mu=1}^{i_t} y_{t\mu\nu}\Big), \quad t = 1, \ldots, \lambda. \tag{3}$$

Since the reproduction axiom is valid in $<R, +, \cdot>$ we can choose $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in R$ such that $y' \in u_1 + \ldots + u_n$, $x \in v_1 + \ldots + v_n$.

Using these expressions we add y' to all members of (2) on the right side and we add x to all members of (3) on the left side, so we obtain for appropriate $\overline{x}$ and $\overline{y}$:

$$\{x_{s-1} + y'\} \cup \{x_s + y'\} \subset \sum_{\nu=1}^{n^{i_s}} \Big(\prod_{\mu=1}^{i_s+1} \overline{x}_{s\mu\nu}\Big), \quad s = 1, \ldots, k, \tag{4}$$

where $\overline{x}_{s\mu\nu} = y' + x_{s\mu\nu}$,

$$\{x + y_{t-1}\} \cup \{x + y_t\} \subset \sum_{\nu=1}^{n^{i_t}} \Big(\prod_{\mu=1}^{i_t+1} \overline{y}_{t\mu\nu}\Big), \quad t = 1, \ldots, \lambda, \tag{5}$$

where $\overline{y}_{t\mu\nu} = x + y_{t\mu\nu}$. From (4) and (5) we take only one element of the sets $x_s + y'$, $x + y_t$ for $s = 1, \ldots, k$, $t = 2, \ldots, \lambda$ and observe that $x' . y' = x_0 . y'$ are in n-fundamental relation with all the elements of set $x . y = x . y_\lambda$.

Therefore we can write the relation (1) in the form

$$F_n(x) + F_n(y) = \{F_n(z)\} \text{ for all } z \in x + y.$$

Proposition 2.7. [13] If $<R, +, \cdot>$ has the identity element 1 w. r. t. $(\cdot)$ and $\{u, v\} \subset x + y$, then $u \underline{n} v$.

Proof. Since the reproduction axiom is satisfied in $<R, +, \cdot>$ then $\exists\ y_1, \ldots, y_n \in R$ such that $y \in y_1 + \ldots + y_n$ and then

$$\begin{aligned} \{u, v\} &\subset x + y \subset x + (y_1 . 1 + y_2 . 1 + \ldots + y_n . 1) \\ &\subset (x + y_1) . (x . 1) + (x + y_2) . (x . 1) + \ldots + (x + y_n) . (x . 1) \\ &= \overline{y}_1 . x + \overline{y}_2 . x + \ldots + \overline{y}_n . x, \end{aligned}$$

where $\bar{y}_i = x + y_i$, $i = 1, \ldots, n$.

Therefore from the definition we obtain $u\hat{n}v$.

It is easily seen that $< F_n, +, \cdot >$ is a hypergroup and $\cdot$ is associative, so, since $+$ is distributive over $.$, $< F_n, +, \cdot >$ is a ring.

Theorem 2.9. [13] Let $< R, +, \cdot >$ be a hyperring with a zero 0 and a unit 1. If $\{u, v\} \subset \prod_{i=1}^{k} x_i$, then $u\hat{n}v$.

Proof. From the reproduction axiom there exist $y_1, y_2, \ldots, y_t \in R$ such that $x_k \in y_1 + \ldots + y_t$, so

$$\{u, v\} \subset \prod_{i=1}^{k-1} x_i \cdot (y_1 + \ldots + y_t)$$

$$\subset \prod_{i=1}^{k-1} x_i \cdot y_1 + \ldots + \prod_{i=1}^{k-1} x_i \cdot y_t$$

$$\subset \prod_{i=1}^{k-1} x_i \cdot y_1 + \ldots + \prod_{i=1}^{k-1} x_i \cdot y_t + \underbrace{0 \,.\, 1^k + 0 \,.\, 1^k + \ldots + 0 \,.\, 1^k}_{n^{k-1} - t \text{ times}},$$

where $1^k = \underbrace{1 \,.\, 1 \ldots 1}_{k \text{ times}}$ so $u\hat{n}v$.

3. n-Hyperring Relation

Definition 3.1. Let $< R, +, \cdot >$ be a hyperring, where $+$ is distributive with respect to $\cdot$. For a given natural number $n \neq 0$ we define in R the n-hyperring relation as follows, which we denote by $\hat{n}$:

(i) $a\hat{n}a$ for all $a \in R$,

(ii) $a\hat{n}b$, $a \neq b$ if we can find elements $x_0, x_1, \ldots, x_k \in R$ where $x_0 = a$, $x_k = b$ and $i_1, i_2, \ldots, i_k > 1$ are natural numbers such that $\{x_{s-1}, x_s\} \subset \sum_{\nu=1}^{n^{i_s - 1}} \left(\prod_{\mu=1}^{i_s} x_{s\mu\nu} \right)$, $s = 1, \ldots, k$ where $x_{s\mu\nu} \in R$.

Remark 3.2. It is very clear that the n-hyperring relation is an equivalence relation.

Proposition 3.3. $F_{\hat{n}}(x) + F_{\hat{n}}(y) = F_{\hat{n}}(z)$, for all $z \in x + y$.

Proof. The proof is analogous to that of Theorem 2.6 except that we add y' to the right side of (2) of Theorem 2.6 using the distributive property of $(+)$ on $(\cdot)$, namely $x + (y \,.\, z) \subset (x + y) \,.\, (x + z)$ to get

$$\{x_{s-1} + y'\} \cup \{x_s + y'\} \subset \prod_{\nu=1}^{n^{i_s}} \Big(\sum_{\mu=1}^{i_s+1} \bar{x}_{s\mu\nu} \Big),$$

where $\bar{x}_{s\mu\nu} = x_{s\mu\nu} + y'$, $s = 1, \ldots, k$, also we add x to the left side of (3) in Theorem 2.6 to get

$$\{x + y_{t-1}\} \cup \{x_s + y_t\} \subset \prod_{\nu=1}^{n^{i_t}} \Big(\sum_{\mu=1}^{i_t+1} \Big) \bar{x}_{t\mu\nu}, \quad t = 1, \ldots, \lambda,$$

where $\bar{y}_{t\mu\nu} = x + y_{t\mu\nu}$. By following the same argument of Theorem 2.6 we get $F_{\hat{n}}(x) + F_{\hat{n}}(y) = F_{\hat{n}}(z)$, for all $z \in x + y$.

Definition 3.4. Let $<R, +, \cdot>$ be a hyperring, it is called strong iff

$$\prod_{i=1}^{k} \Big(\sum_{j=1}^{n} x_{ij} \Big) \subset \sum_{j=1}^{n} \sum_{i=1}^{k-1} x_{ij} \,.\, x_{i+1,j},$$

where $x_{ij} \in R$.

Theorem 3.5. If $<R, +, \cdot>$ is a strong hyperring, then $F_{\hat{n}}(x) \subset F_{n+1}(x)$ for all $x \in R$.

Proof. Let $a \in F_{\hat{n}}(x)$, i.e. $a \hat{n} x$, this imphes that there eexist $x_0, x_1, \ldots, x_k \in R$ with $x_0 = a$ and $x = x_k$ such that

$$\{x_{s-1}, x_s\} \subset \prod_{\nu=1}^{n^{i_s-1}} \Big(\sum_{\mu=1}^{i_s} x_{s\mu\nu} \Big), \quad s = 1, \ldots, k.$$

Since $<R, +, \cdot>$ is a strong hyperring then

$$\prod_{\nu=1}^{n^{i_s-1}} \Big(\sum_{\mu=1}^{i_s} x_{s\mu\nu} \Big) \subset \sum_{\nu=1}^{(n+1)} \Big(\prod_{\mu=1}^{i_s=2} x_{s\mu\nu} \Big).$$

Therefore a and x are in the $n+1$ - fundamental ralation, so $F_{\hat{n}}(x) \subset F_{n+1}(x)$.

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