ON DERIVATION OF SEMIPRIME RINGS

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Summary : In this paper we establish the following result: "If R is a semiprime ring admitting a derivation d such that either (i) $xyx \pm d(xyx) = x^2 y \pm d(x^2 y)$ or (ii) $yxy \pm d(xyx) = xy^2 \pm d(xy^2)$ for all x, $y \in R$, then R must be commutative." Further, if R is prime, then (i) or (ii) need only be assumed for all x, y in some non-zero ideal of R.

YARI ASAL HALKALARIN TÜREVİ HAKKINDA

Özet : Bu çalışmada şu sonuç elde edilmektedir: "R, bir d türevini haiz yarı asal bir halka olsun, öyle ki, her $x, y \in R$ için ya (i) $xyx \pm d(xyx) = x^2 y \pm d(x^2 y)$ veya (ii) $yxy \pm d(xyx) = xy^2 \pm d(xy^2)$ olsun. Bu takdirde R komutatif olmak zorundadır." Bundan başka, R nin asal halka olması durumunda (i) ve (ii) nin R nin sıfır idealden farklı uygun bir idealine ait bütün x, y ler için gerçeklenmesinin yeterli olduğu gösterilmektedir.

INTRODUCTION

Recently, several authors interested in derivations on prime or semiprime rings, and some of the results involved the commutativity of prime and semiprime rings. In this paper, we investigate the commutativity on R of a derivation d satisfying one of the following conditions :

(*) There exists a non-zero ideal I of R such that $xyx \pm d(xyx) = x^2 y \pm \pm d(x^2 y)$ for all x, $y \in I$.

(**) There exists a non-zero ideal I of R such that $xyx \pm d(yxy) = \pm xy^2 \pm \pm d(xy^2)$ for all x, $y \in I$.

MAIN RESULTS

The principle results of this paper are the following:

Theorem 1. Let R be any prime ring admitting a derivation d and satisfying the following: there exists a non-zero ideal I of R such that either $xyx \pm d(xyx) =$

(1991) AMS Subject classification: 16U80

Key words and phrases: Derivations, commutativity, semi-prime rings, prime rings.

$= x^2 y \pm d(x^2 y)$ for all $x, y \in I$, or $xyx \pm d(yxy) = \pm xy^2 \pm d(xy^2)$ for all $x, y \in I$. Then R is commutative.

Theorem 2. Let R be a semiprime ring admitting a derivation d for which either $xyx + d(xyx) = x^2 y + d(x^2 y)$ for all $x, y \in R$ or $xyx - d(xyx) = x^2 y - d(x^2 y)$ for all $x, y \in R$. Then R is commutative.

Theorem 3. Let R be a semiprime ring admitting a derivation d for which either $yxy + d(yxy) = xy^2 + d(xy^2)$ for all $x, y \in R$, or $yxy - d(yxy) = xy^2 - d(xy^2)$ for all $x, y \in R$. Then R is commutative.

Theorem 4. Let R be a semiprime ring with a derivation d satisfying the following: there exists a non-zero ideal I of R such that $xyx \pm d(xyx) = x^2 y \pm \pm d(x^2 y)$ for all $x, y \in I$. Then I is a central ideal.

Theorem 5. Let R be a semiprime ring with a derivation d satisfying the following: there exists a non-zero ideal I of R such that $xyx \pm d(yxy) = \pm xy^2 \pm \pm d(xy^2)$ for all $x, y \in I$. Then I is a central ideal.

Theorem 1 and Theorem 2 are consequences of Theorem 4, and also the proofs of Theorem 1 and Theorem 3 are based on Theorem 5. In this connection we state the following well-known results:

Lamme 1. Let R be a prime ring with a non-zero central ideal. Then R is commutative.

Lemma 2. Let R be a semiprime ring. Then the center of a non-zero ideal is contained in the center of R.

The following lemma is proved in [1]. For the sack of completeness, we prove it.

Lemma 3. Let R be a semiprime ring and let A be a non-zero ideal of R. If a in R centralizes the set $[A, A]$, then a centralizes A.

Proof. Suppose that a centralizes $[A, A]$. Then for all $x, y \in A$, we have

$$a[x, y] = [x, y]\, a. \tag{1}$$

Replace y by xy in (1) to get

$$a[x, xy] = [x, xy]\, a.$$

Thus

$$a(x^2 y - xyx) = (x^2 y - xyx) a,$$
$$ax(xy - yx) = x(xy - yx) a,$$
$$ax[x, y] = x[x, y] a,$$
$$ax[x, y] - xa[x, y] = 0,$$
$$(ax - xa)[x, y] = 0,$$
$$[a, x][x, y] = 0. \qquad (2)$$

Replacing y by xa in (2) gives

$$[a, x][x, xa] = 0,$$
$$[a, x] x [a, x] = 0,$$
$$[a, x] A [a, x] = \{0\}.$$

Since A is an ideal, we have

$$[a, x] AR [a, x] = [a, x] RA [a, x] = \{0\}.$$

Thus

$$[a, x] A = 0$$

or

$$A [a, x] = 0$$

for all $x \in A$. Hence by Theorem 3 of [2], a centralizes A.

Proof of Theorem 4. By hypothesis, there exists a non-zero ideal I such that

$$xyx + d(xyx) = x^2 y + d(x^2 y), \qquad (3)$$
$$xyx - x^2 y = d(x^2 y - xyx),$$
$$x[y, x] = d(x[x, y]), \qquad (4)$$

for all $x, y \in I$.

Further, for all $x, y, z \in I$, we have

$$[x[y, x] - d(x[x, y]), z] = 0,$$
$$x[y, x] z - d(x[x, y]) z = z(x[y, x]) - zd(x[x, y]),$$
$$x[y, x] z - d(x[x, y]) z - (x[x, y]) d(z) = zx[y, x] - d(zx[x, y]),$$
$$x[y, x] z - d(x[x, y]) z - x[x, y] d(z) = zx[y, x] - d(z) x[x, y] - zd(x[x, y]). \qquad (5)$$

Using (4), (5) yields

$$x[x, y] d(z) = d(z) x[x, y],$$

for all x, y, z in I. By Lemma 3, we observe that $d(I)$ centralizes I, and (3) gives that $x[x, y]$ is in the center of I. Thus

$$[x[x, y], z] = 0$$

and

$$[[x, xy], z] = 0$$

for all $z \in I$. Using of the same line of Lemma 1 implies that z centralizes $[I, I]$. Therefore, $[x, y]$ lies in the center of I, for all $x, y \in I$. Also this shows that I is commutative. Thus by Lemma 2 I is in the center of R. Hence R is commutative.

Similary, we can establish that

$$xyx - d(xyx) = x^2 y - d(x^2 y)$$

for all x, y in I.

Proof of Theorem 5. By the hypothesis, there exists a non-zero ideal I such that for all $x, y \in I$,

$$yxy + d(yxy) = xy^2 + d(xy^2),$$
$$yxy - xy^2 = d(xy^2 - yxy),$$
$$(yx - xy)\, y = d((xy - yx)\, y),$$
$$[y, x]\, y = d([x, y]\, y).$$

Repeating the same line as in the proof of Theorem 4, we prove this theorem.

In the same way we can prove that for all $x, y \in I$,

$$yxy - d(yxy) = xy^2 - d(xy^2).$$

Theorem 1 follows from Theorem 4, Theorem 5 and Lemma 1. Also Theorem 2 is direct from Theorem 4 and Theorem 3 is immediate from Theorem 5.

REFERENCES

[1] BELL, H.E. and MARTINDALE, W.S. : *Centralizing mappings of semiprime rings*, Canad. Math. Bull., 30 (1987), 92-101.

[2] BELL, H.E. and NADA, I. : *On some center-like subsets of rings*, Arch. Math., **48** (1987), 381-387.