SOME COMMUTATIVITY RESULTS FOR ONE SIDED s-UNITAL RINGS

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Summary : Let R be an associative ring, $\mathbf{Z}[t]$ is the totality of polynomials in t with coefficients in $\mathbf{Z}$, the ring of integers, and let A be any non-empty subset of R. In this paper, we consider the following ring properties:

(H): For each x, y in R, there exits $f(t) \in t^2\, \mathbf{Z}[t]$ such that $[x - f(x), y] = 0$.

(C): For all x, y in R, there exist $f(t)$, $g(t) \in t^2\, \mathbf{Z}[t]$ such that $[x - f(x), y - g(y)] = 0$.

$(I - A)$: For each x in R either x is central or there exists $f(t) \in t^2\, \mathbf{Z}[t]$ such that $x - f(x) \in A$.

$P(m, n, p, q)$: For each x, y in R, there exists $f(t) \in t^2\, \mathbf{Z}[t]$ such that $[x^m yx^n - x^p f(y) x^q, x] = 0$, where m, n, p, q are fixed non-negative integers.

$P^*(m, n, p, q)$: For each x, y in R, there exist integers $m \geq 0$, $n \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2\, \mathbf{Z}[t]$ such that $[x^m yx^n - x^p f(y) x^q, x] = 0$.

In fact, we prove "If R is a left (resp. right) s-unital ring satisfying $P(m, 0, p, q)$ (resp. $P(0, n, p, q)$), then R is commutative (and conversely)", and "If R is a left (resp. right) s-unital ring satisfying $P^*(m, 0, p, q)$ (resp. $P^*(0, n, p, q)$) and $(I - N(R))$, then R is commutative (and conversely)".

TEK YANLI "s-UNITAL" HALKALAR İÇİN BAZI KOMÜTATİFLİK SONUÇLARI

Özet : R asosyatif bir halka, $\mathbf{Z}[t]$ t nin katsayıları $\mathbf{Z}$ tamsayılar halkasından alınmış bütün polinomlarından oluşan halka, A da R nin boş olmayan herhangi bir alt kümesi olsun. Bu çalışmada

(H): Her $x, y \in R$ için, $[x - f(x), y] = 0$ olacak şekilde $f(t) \in t^2\, \mathbf{Z}[t]$ vardır.

(C): Bütün $x, y \in R$ ler için, $[x - f(x), y - g(y)] = 0$ olacak şekilde $f(t)$, $g(t) \in t^2\, \mathbf{Z}[t]$ vardır.

1990 AMS Subject Classification : 16U80.

Key words and phrases: Commutativity of rings, ring with unity, s-unital rings.

$(I-A)$: Her $x \in R$ için ya x merkeze aittir veya $x - f(x) \in A$ olacak şekilde $f(t) \in t^2$ **Z** $[t]$ vardır.

$P(m, n, p, q)$: Her $x, y \in R$ için, $[x^m yx^n - x^p f(y) x^q, x] = 0$ olacak şekilde $f(t) \in t^2$ **Z** $[t]$ vardır (burada m, n, p, q negatif olmayan sabit tam sayılardır).

$P^*(m, n, p, q)$: Her $x, y \in R$ için, $[x^m yx^n - x^p f(y) x^q, x] = 0$ olacak şekilde $m \geq 0$, $n \geq 0$, $p \geq 0$, $q \geq 0$ tam sayıları ve $f(t) \in t^2$ **Z** $[t]$ vardır. gibi halka özelikleri kullanılarak şunlar ispat edilmektedir:

1) "R, $P(m, 0, p, q)$ $(P(0, n, p, q))$ özeliğini gerçekleyen bir sol (sağ) s-unital halka ise R komütatiftir ve bunun karşıtı da doğrudur",

2) "R, $P^*(m, 0, p, q)$ özeliğini $(P^*(0, n, p, q)$ ve $(I-N(R))$ özeliklerini) gerçekleyen bir sol (sağ) s-unital halka ise R komütatiftir ve bunun karşıtı da doğrudur".

Let R be an associative ring (not necessarily with unity 1). A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called s-unital if $x \in Rx \cap xR$ for all $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a pseudo (resp. pseudo left or pseudo right) identity of F in R. Throughout the paper $Z(R)$ will denote the center of R, $N(R)$ the set of nilpotent elements of R, $C(R)$ the commutator ideal of R, and A a non-empty subset of R. As usual $Z[t]$ is the totality of polynomials in t with coefficients in Z, the ring of integers, and for any $x, y \in R$, $[x, y] = xy - yx$.

By $GF(q)$, we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix}$, and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$ in $(GF(p))_2$ for a prime p.

Now, we consider the following types of rings :

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(\mathrm{i})_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(\mathrm{i})_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $M_\sigma(\mathbf{K}) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| \, a, b \in \mathbf{K} \right\}$, where **K** is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative division ring.

(iv) $S = < 1 > + T$, where T is a non-commutative radical subring of S.

(v) $S = < 1 > + T$, where T is a non-commutative subring of S such that $T[T, T] = [T, T]\ T = 0$.

From the proof of [19, Korollar 1] it can be easily seen that if R is a non-commutative ring with unity 1, then there exists a factorsubring of R, which is of type (i), $(\mathrm{i})_l$, $(\mathrm{i})_r$, (iii), (iv) or (v). This gives the following Meta Theorem, which plays the key role in our subsequent study:

Meta Theorem. Let **P** be a ring property which is inherited by factorsubrings. If no rings of type (i), $(\mathrm{i})_l$, $(\mathrm{i})_r$, (ii), (iii), (iv) or (v) satisfy **P**, then every ring with unity 1 satisfying **P** is commutative.

Next, we consider the following ring properties:

(H) : For each x, y in R, there exists $f(t) \in t^2 Z[t]$ such that $[x-f(x), y]=0$.

(C) : For all x, y in R, there exist $f(t)$, $g(t) \in t^2 Z[t]$ such that $[x - f(x), y - g(y)] = 0$.

$(I - A)$: For each x in R either x is central or there exists $f(t) \in t^2 Z[t]$ such that $x - f(x) \in A$.

$P(m, n, p, q)$: For each x, y in R, there exists $f(t) \in t^2 Z[t]$ such that $[x^m yx^n - x^p f(y) x^q, x] = 0$, where m, n, p, q are fixed non-negative integers.

$P^*(m, n, p, q)$: For each x, y in R, there exist integers $m \geq 0$, $n \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2 Z[t]$ such that $[x^m yx^n - x^p f(y) x^q, x] = 0$.

A well-known theorem of Herstein [10] (signified as Theorem H) asserts that every ring satisfying (H) is commutative. Recently, various authors have studied commutativity of rings satisfying conditions (C), but always under some restrictions (cf. [9], [13] & [15] etc.). More recently, Komatsu et al. [13] investigated the commutativity of rings satisfying the condition $P^*(m, 0, 0, q)$. Further, in a paper [16] Nishinaka established the commutativity of ring R with the conditions $P(m, 0, 0, q)$ and $P(m, 0, p, 0)$. In fact, he proved that a ring R with unity 1 satisfying any one of the conditions $P(m, 0, 0, q)$ and $P(m, 0, p, 0)$ must be commutative. In the present paper, first we shall study the commutativity of rings satisfying $P(m, n, p, q)$ and establish the commutativity of one sided s-unital ring with either of the conditions $P(m, 0, p, q)$ and $P(0, n, p, q)$. We then proceed to investigate the commutativity of rings satisfying $P^*(m, 0, p, q)$ or $P^*(0, n, p, q)$ together with the condition (C). As corollaries to our theorems we shall give several results concerning the commutativity of ring R. The results obtained in sequel generalize [1, Theorem 1.1], [2, Theorems 2&3], [3, Theorems 1&2],

[4, Theorem], [5, Theorem 2], [6, Theorems 1-4], [7, Theorems 4&5], [15 Theorems 2&3 (2)], [16, Theorem 1], [18, Theorem] and [20, Theorem 2 (5)], and thus provide an effective measure to determine the commutativity of R.

We begin with the following lemmas, which are essentially proved in [13] and [15] respectively.

Lemma 1 [13, Corollary 1]. Let R be a ring with unity 1 satisfying (C). If R is non-commutative, then there exists a factorsubring of R, which is of type (i) or (ii).

Lemma 2 [15, Lemma 1]. If R is left s-unital and not right s-unital, then R has a factorsubring of type $(\mathrm{i})_l$.

We pause to remark that the dual of Lemma 2 asserts that if R is right s-unital and not left s-unital, then R has a factorsubring of type $(\mathrm{i})_r$.

The following proposition is an important one from the point of view that it serves as the foundation for our entire discussion.

Proposition 1. Let R be a ring with unity 1. If R satisfies $P(m, n, p, q)$, then there exists no factorsubring of R which is of type (ii), (iii), (iv) or (v).

Proof. Consider the ring $M_\sigma(\mathbb{K})$, a ring of type (ii). Let $x=\begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}$, $(\sigma(\alpha) \neq \alpha)$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$[x^m yx^n - x^p f(y) x^q, x] = - x^m [x, y] x^n = - \alpha^m (\alpha - \sigma(\alpha)) (\sigma(\alpha))^n y \neq 0$$

for every $f(t) \in t^2 Z[t]$. Thus no rings of type (ii) satisfy $P(m, n, p, q)$.

Next, if R is a ring of type (iii), then choose $f(t) \in t^2 Z[t]$ such that

$$[x^{-m} yx^{-n} - x^{-p} f(y) x^{-q}, x^{-1}] = 0.$$

This yields that $[x^{-m} yx^{-n} - x^{-p} f(y) x^{-q}, x] = 0$, that is

$$x^{-m} [x, y] x^{-n} = x^{-p} [x, f(y)] x^{-q}.$$

It follows that

$$x^p [x, y] x^q = x^m [x, f(y)] x^n. \tag{1}$$

Now, choose $g(t) \in t^2 Z[t]$ such that $[x^m f(y) x^n - x^p g(f(y)) x^q, x]=0$. Hence we get

$$x^m [x, f(y)] x^n = x^p [x, g(f(y))] x^q. \tag{2}$$

Comparing of (1) and (2) yields that $x^p [x, y] x^q = x^p [x, h(y)] x^q$, where $h(t)=g(f(t)) \in t^2 Z[t]$. But, since x is unit, $[y-h(y), x]=0$ and by Theorem H, R is commutative, a contradiction. Hence no rings of type (iii) satisfy $P(m, n, p, q)$.

Further, suppose that R has a factorsubring of type (iv). Let $a, b \in T$. Since $1 - a$ is a unit, there exists $f(t) \in t^2 Z[t]$ such that $[a, b - f(b)] = -[1 - a, b - f(b)] = 0$, by above paragraph. Hence, by Theorem H, T is commutative. This is impossible. Hence no rings of type (iv) satisfy $P(m, n, p, q)$.

Finally, suppose that R is of type (v). For each $a, b \in T$, there exists $f(t) \in t^2 Z[t]$ such that

$$[a, b] = (a+1)^m [a, b] (a+1)^n = (a+1)^p [a, f(b)] (a+1)^q = 0.$$

This is a contradiction.

Hence, it proves that no rings of type (ii), (iii), (iv) or (v) satisfy $P(m, n, p, q)$.

Lemma 3. Let R be a ring with unity 1. If for each x, y in R there exists an integer $k = k(x, y) \geq 1$ such that $x^k [x, y] = 0$ or $[x, y] x^k = 0$, then necessarily $[x, y] = 0$.

Proof. Choose an integer $k_1 = k(x, y) \geq 1$ such that $(x+1)^{k_1} [x, y] = 0$. Now, if $N = \max(k, k_1)$, then it follows that $x^N [x, y] = 0$ and $(x+1)^N [x, y] = 0$. We have $[x, y] = \{(x+1) - x\}^{2N+1} [x, y]$. On expanding the expression on right hand side by binomial theorem and using the fact that $x^N [x, y] = 0$ and $(x+1)^N [x, y] = 0$, we get $[x, y] = 0$. Similarly, if $[x, y] x^k = 0$, then using the same techniques, we get the required result.

Lemma 4. Let R be a ring with 1 satisfying any one of the properties $P^*(m, 0, p, q)$ and $P^*(0, n, p, q)$. Then $N(R) \subseteq Z(R)$.

Proof. Property $P^*(m, 0, p, q)$ may be written as $x^m[x, y] - x^p[x, f(y)] x^q = 0$. Let $a \in N(R)$ and x be an arbitrary element of R. Then there exist integers $m_1 = m(x, a) \geq 0$, $p_1 = p(x, a) \geq 0$, $q_1 = q(x, a) \geq 0$ such that $x^{m_1}[x, a] = x^{p_1} [x, f_1(a)] x^{q_1}$ for some $f_1(t) \in t^2 Z[t]$. Similarly, for the pair of elements x, $f_1(a)$, there exist integers $m_2 = m(x, f_1(a)) \geq 0$, $p_2 = p(x, f_1(a)) \geq 0$, $q_2 = q(x, f_1(a)) \geq 0$ such that

$$x^{m_2} [x, f_1(a)] = x^{p_2} [x, f_2(f_1(a))] x^{q_2},$$

for some $f_2(t) \in t^2 Z[t]$, which yields that

$$x^{m_1+m_2} [x, a] = x^{p_1+p_2} [x, f_2(f_1(a))] x^{q_1+q_2}.$$

Thus, it is clear that for an arbitrary k, there exist integers $m_1, m_2, \ldots, m_k \geq 0$, $p_1, p_2, \ldots, p_k \geq 0$, and $q_1, q_2, \ldots, q_k \geq 0$ such that

$$x^{m_1+m_2+\ldots m_k} [x, a] = x^{p_1+p_2+\ldots+p_k} [x, f_k(\ldots f_1(a) \ldots)] x^{q_1+q_2+\ldots+q_k}.$$

But since a is nilpotent, $x^{m_1+m_2+\ldots+m_k} [x, a] = 0$ for sufficiently large k. Hence in view of Lemma 3, we get $[x, a] = 0$ for all x in R. This proves that $N(R) \subseteq Z(R)$.

Using the similar arguments one can establish the result if R satisfies $P^*(0, n, p, q)$.

Following [11], let **P** be a ring property. If **P** is inherited by every subring and every homomorphic image, then **P** is called an h-property. More weakly, if **P** is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then **P** is called an **H**-property.

A ring property **P** such that a ring R has the property **P** if and only if all its finitely generated subrings have **P**, is called an **F**-property.

Proposition 2 [11, Proposition 1]. Let **P** be an **H**-property, and let **P′** be an **F**-property. If every ring R with unity 1 having the property **P** has the property **P′**, then every s-unital ring having **P** has **P′**.

We are now well-equipped to prove the following :

Theorem 1. If R is a left s-unital ring satisfying $P(m, 0, p, q)$, then R is commutative (and conversely).

Proof. Consider the ring of type $(\mathrm{i})_l$. Then

$$[(e_{11}+e_{12})^m\, e_{12}-(e_{11}+e_{12})^p\, f(e_{12})\, (e_{11}+e_{12})^q,\ e_{11}+e_{12}] = -\, e_{12} \neq 0,$$

for all integers $m \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2\, Z[t]$. Accordingly, R has no factorsubrings of type $(\mathrm{i})_l$. Hence by Lemma 2, R is s-unital and in view of Proposition 2, we may assume that R has unity 1.

Combining the above fact with Proposition 1, we see that no rings of type (i), (ii), (iii), (iv) or (v) satisfy the ring property $P(m, 0, p, q)$ and hence by Meta Theorem, R is commutative.

Theorem 2. If R is right s-unital ring satisfying $P(0, n, p, q)$, then R is commutative (and conversely).

Proof. Consider the ring of type $(\mathrm{i})_r$. Then

$$[e_{12}\, e_{22}^n - e_{22}^p\, f(e_{12})\, e_{22}^q,\ e_{22}] = e_{12} \neq 0,$$

for all integers $n \geq 0$, $p \geq 0$, $q \geq 0$ and $f(t) \in t^2\, Z[t]$. Thus, R has no factorsubrings of type $(\mathrm{i})_r$ and by the dual of Lemma 2, R is s-unital. Now, using the same arguments as used in the proof of Theorem 1, we get the required result.

As corollaries to our theorems we have the following results improving [1, Theorem 1.1], [4, Theorem], [5, Theorem 2 (iii)], [6, Theorems 1-4], [7, Theorems 4&5], [15, Corollary 2 (3)], [18, Theorem] and [20, Theorem 2 (5)]. Also, note that Theorem 1 generalizes the results proved in [15, Theorem 2] and [16, Theorem 1].

Corollary 1. Let m, p and q be fixed non-negative integers, and let R be a left s-unital ring. If for each x, y in R, there exists an integer $s=s(x, y)>1$ such that $[x^m y - x^p y^s x^q, x] = 0$, then R is commutative (and conversely).

Corollary 2. Let n, p, and q be fixed non-negative integers, and let R be a right s-unital ring. If for each x, y in R, there exists an integer $s=s(x, y)>1$ such that $[yx^n - x^p y^s x^q, x] = 0$, then R is commutative (and conversely).

Theorem 3. Let R be a left s-unital ring satisfying $P^*(m, 0, p, q)$ and $(I - N(R))$. Then R is commutative (and conversely).

Proof. It is easy to see that the arguments given in the first paragraph of the proof of Theorem 1 are still valid in the present situation. So we assume henceforth that R has unity 1 and no rings of type (i) satisfy the condition $P^*(m, 0, p, q)$. Also, if R is a ring of type (ii), then choose $x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}$ $(\sigma(\alpha) \neq \alpha)$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, to get

$$[x^m y - x^p f(y) x^q, x] = - x^m[x, y] = - \alpha^m (\alpha - \sigma(\alpha)) y \neq 0,$$

for every $f(t) \in t^2 Z[t]$. Thus no rings of type (ii) satisfy $P^*(m, 0, p, q)$. Since $N(R) \subseteq Z(R)$ by Lemma 4, it is straightforward to see that R satisfies (C). Hence, in view of Lemma 1, R is commutative.

The following theorem can also be proved on the same lines as above, employing necessary variations.

Theorem 4. Let R be a right s-unital ring satisfying $P^*(0, n, p, q)$ and $(I - N(R))$. Then R is commutative (and conversely).

As an immediate consequence of the above theorems, we obtain the following results improving [2, Theorem 2], [5, Theorem 2 (iii)], [15, Corollary 2 (2)] and [20, Theorem 2 (4)].

Corollary 3. Let R be a left s-unital ring. Suppose that for each x, y in R, there exist integers $m \geq 0$, $p \geq 0$, $q \geq 0$ and $s > 1$ such that $[x^m y - x^p y^s x^q, x] = 0$ and for each x in R either x is central or there exists $f(t) \in t^2 Z[t]$ such that $x - f(x) \in N(R)$. Then R is commutative (and conversely).

Corollary 4. Let R be a right s-unital ring. Suppose that for each x, y in R, there exist integers $n \geq 0$, $p \geq 0$, $q \geq 0$ and $s > 1$ such that $[yx^n - x^p y^s x^q, x] = 0$ and for each x in R either x is central or there exists $f(t) \in t^2 Z[t]$ such that $x - f(x) \in N(R)$. Then R is commutative (and conversely).

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