ERROR IN DEMARTRES'S PAPER AND THE CORRECT RESULT

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Summary : Demartres has considered the problem of determining isothermic Weingarten surfaces in [2], and proved that all the isothermic Weingarten surfaces consist of the surfaces of revolution, surfaces of constant mean curvature, cones, cylinders and isothermic helicoids.

In our present paper, the same problem is considered again, and the following results are obtained:

1. It is shown that the Demartres's result is incomplete, and all the isothermic Weingarten surfaces are completed by a newly found class of isothermic Weingarten surface.

2. It is proved that the mean curvatures of the new isothermic Weingarten surfaces and the isothermic helicoids satisfy the same ordinary differential equation of order three.

3. It is also shown that all the isothermic Weingarten surfaces, except some of the surfaces of constant mean curvature, can be isometrically mapped to the surface of revolution.

DEMARTRES'IN ÇALIŞMASINDAKİ HATA VE DOĞRU SONUÇ

Özet : Demartres [2]'deki çalışmasında izotermik Weingarten Yüzeylerinin belirtilmesi problenini ele almış ve tüm izotermik Weingarten yüzeylerinin dönel yüzeyler, ortalama eğriliği sabit olan yüzeyler, koniler, silindirler ve izotermik helikoidlerden ibaret olduğunu göstermiştir.

Bu çalışmamızda aynı problem yeniden ele alınmış ve aşağıdaki sonuçlar elde edilmiştir:

1. Demartres'ın bulduğu sonucun eksik olduğu gösterilmiş ve tüm izotermik Weingarten yüzeyleri yeni bulduğumuz bir izotermik Weingarten yüzey sınıfı ile tamamlanmıştır.

2. Yeni bulunan izotermik Weingarten yüzeylerinin ve izotermik helikoidlerin ortalama eğriliklerinin 3. mertebeden aynı bir adi diferansiyel denklemi sağladıkları gösterilmiştir.

3. Bazı sabit ortalama eğrilikli yüzeyler dışındaki bütün izotermik Weingarten yüzeylerinin dönel yüzeye izometrik olarak tasvir edilebileceği de gösterilmiştir.

1. INTRODUCTION

1.1. We have mentioned from the Demartres's paper in [1]. Demartres considered, in his paper [2], the problem of determining of the isothermic Weingarten (*W*-) surfaces and showed that there is not another isothermic W-surface except surfaces of revolution, surfaces of constant mean curvature, cones, cylinders, and isothermic helicoids. But this result is not correct.

There exist isothermic W-surfaces apart from surfaces above, and they have more generality than isothermic helicoids. We call these surfaces "new isothermic W-surfaces".

In this paper, we considered the problem again, which needs complicated calculations and investigations in detail, and obtained the following results:

(1) The mean curvatures of the new isothermic W-surfaces and of the isothermic helicoids satisfy the same ordinary differential equation of order three. Therefore, the quantities of the new isothermic W-surfaces can be directly written by means of the function of mean curvature to be found for the isothermic helicoids.

(2) All the isothermic W-surfaces, except the surfaces of constant mean curvature, can be isometrically mapped to the surfaces of revolution (As known some surfaces of constant mean curvature can not be isometrically mapped to any surface of revolution).

1.2. We assume that the surfaces under consideration are real, and sufficiently differentiable. The coefficients of first and the second fundamental form will be denoted by E, F, G and L, M, N respectively, and the definitions

$$K = r \cdot r^*, \quad H = \frac{r + r^*}{2}, \quad J = \frac{r - r^*}{2}$$

will be used where r and r^* are the principal curvatures. We show the geodesic curvatures of the lines of curvature u and v by $-q$ and q^* respectively. Therefore, in the parameter system which consists of the lines of curvature,

$$F = M = 0, \quad r = \frac{L}{E} = H + J, \quad r^* = \frac{N}{G} = H - J$$
$$q = \frac{e_v}{eg}, \quad q^* = \frac{g_u}{eg}, \quad K = H^2 - J^2 \tag{1.1}$$

can he written, where $e = +\sqrt{E}$, $g = +\sqrt{G}$, $J \neq 0$, $\dfrac{\partial f}{\partial u} = f_u$.

We denote arbitrary constants by a_0, a_1, A_0, B_2 ..., and a letter with the subindex will always denote an arbitrary constant.

2. The system of Mainardi-Codazzi equations and their solutions

2.1. Assume that the surface under consideration is an isothermic W-surface. So we have

$$r = r(T),\ r^* = r^*(T),\ (K = K(T),\ H = H(T),\ J = J(T),\ T = T(u, v)), \quad (2.1)$$

and since the coordinate lines are the lines of curvature, we can take

$$E = G\,(= e^2),\ (F = M = 0). \quad (2.2)$$

2.2. By means of the conditions (2.1) and (2.2), the Mainardi-Codazzi compatibility equations can be written in the form

$$[H'(T) + J'(T)]\,T_v = -\frac{E_v}{E}J,\ [H'(T) - J'(T)]\,T_u = \frac{E_u}{E}J \quad (2.3)$$

or

$$\left(\int \frac{dH}{J}\right)_v = -(\ln E\,|J|)_v,\ \left(\int \frac{dH}{J}\right)_u = (\ln E\,|J|)_u. \quad (2.4)$$

Here we exclude the surfaces of revolution, surfaces of constant mean curvature, and cones, and cylinders from our consideration. This means that

$$T_u \cdot T_v \cdot r'(T) \cdot r^{*\prime}(T) \cdot [r'(T) + r^{*\prime}(T)] \neq 0. \quad (2.5)$$

For: (i) If the surface is a surface of revolution then $T_u \cdot T_v = 0$; conversely if $T_u \cdot T_v = 0$ (say $T_v = 0$, $T_u \neq 0$) then from (2.1) we obtain $r_v = r^*_v = 0$, so the surface is rotational [3, p. 59]. (ii) If $r'(T) = 0$ (or $r^{*\prime}(T) = 0$), then from (2.3) we get $E = E(u)$, and from the Gauss equation to be written by this $E(u)$ in the case of $r = r_0 = \text{const.} \neq 0$, we obtain $r^* = r^*(u)$ (or $r = r(v)$); so in this case the surface is a rotational Dupin eyelid, i.e., it is torus [3, pp.59, 61]. Therefore, if the surface is not rotational then $T_u \cdot T_v \cdot r'(T) \cdot r^{*\prime}(T) \neq 0$. In the case of $r = r_0 = 0$ (the case of developable surface), we know that the surface is a cone or a cylinder [3, p. 54]. (iii) If $r'(T) + r^{*\prime}(T) = 0$, then $H = \text{const.}$ and vice versa. Therefore, the condition (2.5) is the necessary and sufficient condition to exclude the mentioned surfaces from our consideration.

Since $H' \neq 0$, from (2.4)

$$\int \frac{dH}{J} = -\ln E\,|J| + 2U(u),\quad \int \frac{dH}{J} = \ln E\,|J| + 2V(v)$$

or

$$\int \frac{dH}{J} = U(u) + V(v),\ \ln E\,|J| = U(u) - V(v) \quad (2.6)$$

can be directly written. This means that $T = T(U + V)$. From this result, since $T' \neq 0$, by means of (2.5), we obtain

$$r = r(t),\ r^* = r^*(t),\ [H = H(t),\ J = J(t)] \tag{2.7}$$

where

$$U(u) + V(v) = t. \tag{2.8}$$

Then the condition (2.5) reduces to the condition

$$r' \cdot r^{*\prime} \cdot H' \cdot U' \cdot V' \neq 0; \tag{2.9}$$

so the results (2.6) are replaced by the results

$$H'(t) = J(t),\ \ln E\,|J| = U(u) - V(v),\ (t = U(u) + V(v),\ H' \neq 0). \tag{2.10}$$

It is easy to verify that these results consist of the solutions of the Mainardi-Codazzi equations (2.4) under the condition (2.9).

3. The Gauss Equation

3.1. The important part of the problem is the problem of satisfying the Gauss equation. Since $U'\,V' \neq 0$ by means of (2.9), we can make the scale transformation

$$x = U(u),\ y = V(v),\ (t = x + y). \tag{3.1}$$

In the parameter system x, y the coefficients of the first fundamental form, are obtained as

$$E^0\left(= \frac{E}{a(x)}\right) = \frac{e^{x-y}}{a\,|J|},\quad G^0 = \frac{e^{x-y}}{b\,|J|},\quad (F_0 = 0) \tag{3.2}$$

where

$$a(x) = U'^2(u),\quad b(y) = V'^2(v). \tag{3.3}$$

The Gauss equation

$$K = H^2 - J^2 = -\frac{1}{2e^0 g^0}\left[\left(\frac{E^0_y}{e^0 g^0}\right)_y + \left(\frac{G^0_x}{e^0 g^0}\right)_x\right]$$

reduces, by the values (3.2), to the equation

$$\frac{2(J^2 - H^2)}{|J|} = e^{y-x}\left[\frac{a'}{2}\left(1 - \frac{J'}{J}\right) - \frac{b'}{2}\left(1 + \frac{J'}{J}\right) - (a + b)\left(\frac{J'}{J}\right)'\right]. \tag{3.4}$$

Since $t = x + y$ and $J(t) = H'(t)$, it is seen that the equation (3.4) is a functional-differential equation to be satisfied by the functions $H(t)$, $a(x)$, and $b(y)$.

The left-hand side of the equation (3.4) depends only on the variable $t = x + y$. Therefore, as independent variables, let us take x and t instead of x and y for a while. So $y = t - x$, and for a function $f(y) = f(t - x)$, $f_x(y) = -f'(y)$ is obtained. Now, let us differentiate both sides of the equation (3.4) with respect to x. Then, the left-hand side becomes zero and from the derivative of the right-hand side, beacuse of $f_x(y) = -f'(y)$,

$$2\left(\frac{J'}{J}\right)'[2a-a'+2b-b']+\left(\frac{J'}{J}\right)[(2a-a')'+(2b+b')']+(2b+b')'-(2a-a')'=0$$

is obtained.

For abbreviation, let us make the following definitions:

$$\frac{J'}{J}=h(t), \quad 2a-a'=\alpha(x), \quad 2b+b'=\beta(y). \tag{3.5}$$

Therefore, we obtain the main equation as

$$2(\alpha+\beta)\,h'(t)+(\alpha'+\beta')\,h(t)+\beta'-\alpha'=0. \tag{3.6}$$

It is clear that

$$\alpha=-\beta=2c_0, \quad (\alpha'=\beta'=0), \quad h=h(t) \tag{3.7}$$

is a trivial solution of the equation (3.6). It is an expected result that the solution (3.7) is the unique solution of the functional-differential equation (3.6), which satisfies the equation (3.4) as well.

In fact, if the equation (3.6) is satisfied for any special value of the function $h(t)\left(=\frac{J'}{J}=\frac{H''}{H'}\right)$, i.e., if the function $h(t)$ satisfies a condition, it is not an expected result that this condition agrees with the equation (3.4), which reduces to a differential equation, whose right-hand side is a function of the variable t, of third-order in $H(t)$.

Now let us prove this claim:

First, if $\alpha+\beta\neq 0$, i.e., if the equation (3.6) has a solution different from the solution (3.7), then we show that

$$h'\cdot\alpha'\cdot\beta'\neq 0. \tag{3.8}$$

(a) If $h'=0$ (so $h=h_0$, $J=h_1\,e^{h_0 t}$) and $\alpha'\beta'\neq 0$, then from (3.6) we get $h_0^2-1\neq 0$ and so from (3.6) and (3.5) we obtain

$$a'=2a_0\,e^{2x}+\frac{c_0}{2(1-h_0)}, \quad b'=-2b_0\,e^{-2y}+\frac{c_0}{2(1+h_0)}.$$

By these values, the right-hand side of (3.4) takes the form $a_0\,e^t(1-h_0)+$ $+b_0\,e^{-t}(1+h_0)$; so it can not be equal to the left-hand side, which is obtained by the value H to be found from (2.10), no matter how the constants are chosen $(\alpha'\,\beta'(h_0^2-\mathrm{I})\neq 0)$. So if $h'=0$, then $\alpha'=0$ or $\beta'=0$. Now let us assume that $h'=\alpha'=0$, $\beta'\neq 0$ (or $h'=\beta'=0$, $\alpha'\neq 0$). Then from (3.6) we have $h=-1$ (or $h=1$) and the right-hand side of (3.4) becomes $2a_0\,e^t\left(a=a_0\,e^{2x}-\frac{\alpha_0}{2}\right)$.
Since $J=k_0\,e^{-t}$ and $H=-k_0\,e^{-t}+k_1$, the equation (3.4) is satisfed for only

$a_0 = k_1 = 0$. But then $K = r \cdot r^* = 0$, which contradicts to the assumption (2.9). So $\beta' = 0$.

(b) If $\alpha' = 0$ $(\alpha = \alpha_0)$ and $h' \beta' \neq 0$ (or $\beta' = 0$ $(\beta = \beta_0)$, $h' \alpha' \neq 0$), by the similar calculations, first

$$\beta = b_1 e^{-2c_0 y} - \alpha_0 , \quad h + 1 = c_1 e^{c_0 t}, \quad a = a_0 e^{2x} + \frac{\alpha_0}{2}, \quad (c_0 \neq 0)$$

are found. By this value of β from (3.5), in case of $c_0 \neq 1$, $b = b_0 e^{-2y} - \dfrac{b_1 e^{-2c_0 y}}{2c_0 - 2} - \dfrac{\alpha_0}{2}$ is found, in case of $c_0 = 1$, $b = (b_1 y + b_0) e^{-2y} - \dfrac{\alpha_0}{2}$. Then the right-hand side of (3.4), for $c_0 \neq 1$, becomes $2a_0 e^t (1 - c_0 c_1 e^{c_0 t})$, for $c_0 = 1$, $2a_0 e^t (1 - c_1 e^t) - \dfrac{b_1 c_1}{2}$. Since $h = \dfrac{J'}{J}$ and $H' = J$, the left-hand side of (3.4) will be a complicated function including an exponential function of $e^{c_0 t}$. This means that the left-hand side will not be equal to the right-hand side. So $h' \beta' = 0$. If $\alpha' = 0$ and $\alpha + \beta \neq 0$, then it is easly shown that $h' = \beta' = 0$.

Thus, in case of $\alpha' \beta' h' = 0$ and $\alpha + \beta \neq 0$, we showed that $\alpha' = \beta' = h' = 0$. Then the equation (3.6) is satisfied. But, using the equations (3.5) and (2.10) and carrying out the similar operations above, it can be shown that the equation (3.4) can not be satisfied under the condition (2.9).

Therefore, we proved that, for the solutions which are different from the solution (3.7), the condition (3.8) is valid.

3.2. Now we pass the general investigation. Above method, that is the method of taking x and t as independent variables and differentiation with respect to x, can be applied to the equation (3.6) infinitely many.

By the first two applications of the method, we find the equations

$$\begin{aligned} &2(\alpha' - \beta') h'(t) + (\alpha'' - \beta'') h(t) - \alpha'' - \beta'' = 0, \\ &2(\alpha'' + \beta'') h'(t) + (\alpha''' + \beta''') h(t) - \alpha''' + \beta''' = 0. \end{aligned} \tag{3.9}$$

By eliminating h and h' $(h \cdot h' \neq 0)$ from (3.6) and (3.9), since $\alpha' \beta' \neq 0$,

$$\begin{vmatrix} \alpha + \beta & 1 & 1 \\ \alpha' - \beta' & \dfrac{\alpha''}{\alpha'} & -\dfrac{\beta''}{\beta'} \\ \alpha'' + \beta'' & \dfrac{\alpha'''}{\alpha'} & \dfrac{\beta'''}{\beta'} \end{vmatrix} = 0 \quad (\alpha' \cdot \beta' \neq 0) \tag{3.10}$$

is found. It is seen that the eight ones of the nine determinants, which are obtained by successive differentiation with respect to x and y of the determinant (3.10), are zero. Since $\alpha + \beta \neq 0$, from the ninth determinant

$$\left(\frac{\alpha'''}{\alpha'}\right)' = 2c_0\left(\frac{\alpha''}{\alpha'}\right)', \quad \left(\frac{\beta'''}{\beta'}\right)' = -2c_0\left(\frac{\beta''}{\beta'}\right)'$$

are obtained and so

$$\alpha'' = 2c_0\,\alpha' + a_1\,\alpha + a_2\,, \quad \beta'' = -2c_0\,\beta' + b_1\,\beta + b_2 \tag{3.11}$$

are found. Substituting (3.11) in (3.10), we obtain the following results:

a) The case $c_0 \neq 0$. In this case, we find first

$$\frac{a_1\,\alpha + a_2}{\alpha'}(c_1\,\alpha + c_2) + 2c_0\,c_1\,\alpha = \frac{b_1\,\beta + b_2}{\beta'}(c_1\,\beta - c_2) - 2c_0\,c_1\,\beta = A_0$$

and from (3.11) we have $c_1 = c_2 = 0$, that is,

$$a_1 - b_1 = a_2 + b_2 = 0, \quad (A_0 = 0) \tag{3.12}$$

where

$$a_1 - b_1 = c_1\,, \quad a_2 + b_2 = c_2\,, \quad (\alpha' \cdot \beta' \neq 0). \tag{3.13}$$

Therefore,

$$\alpha'' = 2c_0\,\alpha' + a_1\,\alpha + a_2\,, \quad \beta'' = -2c_0\,\beta' + a_1\,\beta - a_2\,, \quad (c_0 \neq 0). \tag{3.14}$$

It is easily seen that the equation (3.10) is satisfied by (3.14).

b) The case $c_0 = 0$. From (3.11) it can be directly written

$$\alpha'^2 = a_1\,\alpha^2 + 2a_2\,\alpha + a_3\,, \quad \beta'^2 = b_1\,\beta^2 + 2b_2\,\beta + b_3 \tag{3.15}$$

and from (3.10)

$$c_1\left(\frac{a_2\,\alpha + a_3}{\alpha'}\right) + c_2\left(\frac{a_1\,\alpha + a_2}{\alpha'}\right) = c_1\left(\frac{b_2\,\beta + b_3}{\beta'}\right) - c_2\left(\frac{b_1\,\beta + b_2}{\beta'}\right) = B_0 \tag{3.16}$$

are obtained. Since the solution (3.14) satisfies (3.10) for $c_0 = 0$ also, it is natural that (3.16) is satisfied by the conditions (3.12). If the conditions (3.12) are not valid, then from (3.16), we obtain

$$\alpha' = a_0\,\alpha + a_1\,, \quad \beta' = b_0\,\beta + b_1\,. \tag{3.17}$$

Then the right-hand sides of (3.15) take the perfect squared form. By substituting (3.17) in (3.10)

$$(a_0 + b_0)\,(a_0\,b_1 + b_0\,a_1) = 0$$

is found. From this:

b1) If $a_0\,b_0 = 0\,(\alpha' \cdot \beta' \neq 0)$, then $a_0 = b_0 = 0$ are obtained, so we have

$$\alpha' = a_1\,, \quad \beta' = b_1 \quad (a_1 \cdot b_1 \neq 0); \tag{3.18}$$

b2) if $a_0 b_0 \neq 0$ and $a_0 + b_0 = 0$, then we have

$$\alpha' = a_0 \alpha + a_1, \quad \beta' = - a_0 \beta + b_1; \tag{3.19}$$

b3) if $a_0 b_0 (a_0 + b_0) \neq 0$, then we have

$$\alpha' = a_0 (\alpha + A_0), \quad \beta' = b_0 (\beta - A_0), \quad [a_0 b_0 (a_0 + b_0) \neq 0]. \tag{3.20}$$

Therefore, we have shown that the solutions satisfying (3.10) are the solutions (3.14), (3.18), (3.19) and (3.20).

3.3. Here we will show that none of the solutions above satisfy the equation (3.4).

A) Let us consider the solution (3.18) and use it in (3.6). Then, for $a_1 \neq b_1$, $h'(t) = 0$ is obtained and this contradicts the condition (3.8). For $a_1 = b_1$, from (3.6) and (3.5)

$$(\alpha = a_1 x + a_2, \quad \beta = a_1 y + b_2, \quad c_2 = a_2 + b_2, \quad a_1 \neq 0),$$

$$h(t) = \frac{h_0}{a_1 t + c_2}, \quad a(x) = a_0 e^{2x} + \frac{a_1}{2} x + \frac{a_1}{4} + \frac{a_2}{2},$$

$$b(y) = b_0 e^{-2y} + \frac{a_1}{2} y - \frac{a_1}{4} + \frac{b_2}{2}$$

are obtained. By these values, the right-hand side of the equation (3.4) takes the form

$$a_0 e^t \left(1 - \frac{h_0}{a_1 t + c_2} + \frac{h_0 a_1}{(a_1 t + c_2)^2}\right) + b_0 e^{-t} \left(1 + \frac{h_0}{a_1 t + c_2} + \frac{h_0 a_1}{(a_1 t + c_2)^2}\right),$$

and since at least one of the a_0 and b_0 is not zero, because of the condition (2.9), this right-hand side depends on e^t also. From (2.10) and (3.5) $H' = J = k_0 (a_1 t + c_2)^{h_0/a_1}$ is obtained. Thus, the left-hand side of (3.4), for $h_0 + a_1 = 0$, has the various powers and the logarithm of $(a_1 t + c_2)$; for $h_0 + a_1 \neq 0$, has only various powers of it. So the two sides of (3.4) can not be equal to each other.

B) If the values

$$\alpha = A_1 e^{a_0 x} - \frac{a_1}{a_0}, \quad \beta = B_1 e^{-a_0 y} + \frac{b_1}{a_0}, \quad (A_1 B_1 \neq 0),$$

obtained from (3.19) are used in (3.6), then $(h' \neq 0)$ $a_1 = b_1$ and

$$h(t) = \frac{A_1 e^{a_0 t} - B_1 + h_0 e^{\frac{a_0}{2} t}}{A_1 e^{a_0 t} + B_1}, \quad (A_1 \cdot B_1 \neq 0) \tag{3.21}$$

are obtained. By these values of α, β and h, the equation (3.6) is satisfied.

From (3.5), for a and b,

B1) if $a_2 \neq 2$, then we find

$$a = A_2 e^{2x} - \frac{A_1}{a_0 - 2} e^{a_0 x} - \frac{a_1}{2a_0}, \quad b = B_2 e^{-2y} - \frac{B_1}{a_0 - 2} e^{-a_0 y} + \frac{a_1}{2a_0};$$

B2) if $a_0 = 2$, then we find

$$a = (A_2 - A_1 x) e^{2x} - \frac{a_1}{4}, \quad b = (B_2 + B_1 y) e^{-2y} + \frac{a_1}{4}.$$

In the case of (B1), the right-hand side of the equation (3.4), by the value of $h(t)$ in (3.21) is obtained as

$$[(A_2 e^t + B_2 e^{-t})(1 - h') - (A_2 e^t - B_2 e^{-t}) h].$$

It is seen that this expression is a function of e^t and $e^{a_0 t}$. But, since $h(t) = -\frac{J'}{J}$, by means of (3.21), the function $J(t)$, no matter h_0 is equal to zero or not, has also the logarithm of the function $\left(A_1 e^{\frac{a_0}{2} t} + B_1 e^{-\frac{a_0}{2} t}\right)^{\frac{2}{a_0}}$, $(A_1 \cdot B_1 \neq 0)$. This means that the equation (3.4) is not satisfied in this case. In the case of (B2) the right-hand side of the equation (3.4) becomes

$$\left[\frac{2e^t (A_1 e^{2t} - B_1 + h_0 e^t)(A_1 B_2 - A_2 B_1 + A_1 B_1 t)}{(A_1 e^{2t} + B_1)^2} + \frac{h_0 (A_1 e^{2t} - B_1) - 4A_1 B_1 e^t}{2(A_1 e^{2t} + B_1)}\right],$$

i.e., it is an algebraic function of e^t and t. But the left-hand side has the logarithm of $(A_1 e^t + B_1 e^{-t})$. So the equation (3.4) can not be satisfied in the case of (B2).

C) If the values $\alpha = A_1 e^{a_0 x} - A_0$, $\beta = B_1 e^{b_0 y} + A_0$, obtained from (3.20), are used in (3.6), then $a_0 + b_0 = 0$. This means that the solution (3.19), which is investigated in the case of (B), is obtained.

D) Finally, let us consider the solution (3.14).

D1) For $a_1 = 0$, since the case $a_2 = 0$ corresponds to the case of (B), we can assume that $a_2 \neq 0$. Then, using the values

$$\alpha = A_1 e^{2c_0 x} - \frac{a_2}{2c_0} x + A_2, \quad \beta = B_1 e^{-2c_0 y} - \frac{a_2}{2c_0} y + B_2$$

in (3.6), $(c_0 a_2 \neq 0)$, $A_1 = B_1 = 0$ are obtained, which corresponds to the case (A).

D2) If $a_1 \neq 0$, $c_0^2 + a_1 \neq 0$, then from (3.14)

$$\alpha = A_1 e^{a_0 x} + A_2 e^{b_0 x} - \frac{a_2}{a_1}, \quad \beta = B_1 e^{-a_0 y} + B_2 e^{-b_0 y} + \frac{a_2}{a_1},$$

are obtained, where $(a_0 - b_0) a_0 b_0 \neq 0$. By these values, from (3.6), $A_2 = B_2 = 0$ (or $A_1 = B_1 = 0$) are obtained. Thus, we have the case of (B).

D3) If $c_0^2 + a_1 = 0$, $(c_0 a_1 \neq 0)$, then from (3.14)

$$\alpha = (A_1 x + A_2)\, e^{c_0 x} - \frac{a_2}{a_1}, \quad \beta = (B_1 y + B_2)\, e^{-c_0 y} + \frac{a_2}{a_1}$$

are obtained. Using this result in (3.6) we get $A_1 = B_1 = 0$ which corresponds to the case of (B). Hence, under the condition (2.9), the unique solution which satisfies also the Gauss equation (3.4) is the solution (3.7).

4. Surfaces corresponding to the solution (3.7)

4.1. By means of (3.5), from the solution (3.7), we get

$$a(x) = a_0\, e^{2x} + c_0\,, \quad b(y) = b_0\, e^{-2y} - c_0\,. \tag{4.1}$$

Since we have assumed that the surface under the consideration are real, because of the definitions (3.3) and the condition (2.9), a and b must be positive. Hence, according to (4.1), the constants a_0 and b_0 can be neither negative nor zero at the same time. Accordingly, it is easily seen that the equation (3.4) reduces to the equation

$$\begin{aligned} 2\varepsilon\, H'(H'^2 - H^2) &= a_0\, e^t\, [H'^2 + H''^2 - H'(H''' + H'')] + \\ &+ b_0\, [H'^2 + H''^2 - H'(H''' - H'')] \quad (\varepsilon = \operatorname{Sgn}(J)) \end{aligned} \tag{4.2}$$

which does not depend on the constant c_0. Thus, the problem of determining of the quantities of our surface is reduced to the problem of finding of the solution $H(t)$ of the equation (4.2).

For, if $H(t)$ is known, then, from $J(t)=H'(t)$, $J(t)$ is found (and so $r(t)$ and $r^*(t)$). By the aid of the value of this $J(t)$ and (4.1), from (3.2) E^0 and G^0 $(F^0=0)$ are determined, and from (1.1) $L^0=r(t)\,E^0$, $N^0=r^*(t)\,G^0$, $M^0=0$ can be directly written.

By the condition (2.9), we left the surfaces of revolution, the surfaces of constant mean curvature, and cones and cylinders aside.

So, according to [2] all the surfaces which will be found by the values $a(x)$ and $b(y)$ to be obtained from (4.1), must consist of only isothermic helicoids. In the following we will show that this is not correct and the surfaces corresponding to the case $c_0 \neq 0$ are not the isothermic helicoids, and they form a new class of surfaces.

4.2. Now we will give an important criterium which determines a W-surface whose quantities are given in the curvature lines parameters, whether it is helicoid or not.

Theorem. A necessary and sufficient condition for a non-developable W-surface, whose the principal curvatures are $r(T)$ and $r^*(T)$ $(T=T\ (u, v))$ and the geodesic curvatures of the lines of curvature are $-q$ and q^*, to be a helicoid is that

$$q = q(T), \quad q^* = q^*(T). \tag{4.3}$$

Proof. Let φ be the angle between the constant curvature curves (the family of the curve $T = \text{const.}$) and the family of the lines of curvature $v = \text{const.}$ $(2\varphi = \theta)$. Let k, $-p$ and τ be, respectively, the normal curvature, the geodesic curvature, and the geodesic torsion of the family $T = \text{const.}$ and k^*, p^* and $-\tau$ be those of the orthogonal trajectories.

We denote the derivatives (invariant) in the directions of the lines of curvature $v = \text{const.}$ and $u = \text{const.}$ by the subindices **1** and **2**; and for the lines $T = \text{const.}$ and the orthogonal trajectories, they will be denoted by the indices **1*** and **2*** respectively. Accordingly, we know that

$$k = H + J\cos\theta, \quad k^* = H - J\cos\theta, \quad \tau = -J\sin\theta,$$

$$-p = (\varphi_1 - q)\cos\varphi + (\varphi_2 + q^*)\sin\varphi, \quad p^* = (q - \varphi_1)\sin\varphi + (q^* + \varphi_2)\cos\varphi \tag{4.4}$$

and for a general point function $P(u, v)$

$$P_{1^*} = P_1\cos\varphi + P_2\sin\varphi, \quad P_{2^*} = -P_1\sin\varphi + P_2\cos\varphi \tag{4.5}$$

can be written.

Now let us assume that our surface is a non-developable helicoid. Hence, in addition to $H = H(T)$ and $J = J(T)$ which are the conditions for a W-surface, the conditions $p^* = 0$, $p = p(T)$, $\tau = \tau(T)$, and $k = k(T)$ must be satisfied [3, p.85]; therefore (4.4) implies $\varphi = \varphi(T)$. It is clear that a necessary and sufficient condition for a point function $P(u, v)$ to be the function of only $T(u, v)$, in the parameter system the lines $T(u, v) = \text{const.}$ and the orthogonal trajectories $S(u, v) = \text{const.}$, is that

$$P_{1^*} = 0, \quad \left(P_S(u, v) = 0, \quad P_{1^*} = \frac{P_S}{\sqrt{E^*}}\right).$$

Hence $H_{1^*} = J_{1^*} = \varphi_{1^*} = 0$, that is,

$$H_1\cos\varphi + H_2\sin\varphi = 0, \quad J_1\cos\varphi + J_2\sin\varphi = 0, \quad \varphi_1\cos\varphi + \varphi_2\sin\varphi = 0. \tag{4.6}$$

In addition to these equations, we have the Mainardi-Codazzi equations

$$(H + J)_2 = 2qJ, \quad (H - J)_1 = 2q^*J. \tag{4.7}$$

Here we can assume that $H'J'\varphi' \neq 0$. For, because of the condition (2.9), $H' \neq 0$; for $J' = 0$, the equation (3.4) to be written by the value $H = J_0\, t + t_0$ which will be obtained from (2.10), and by the values a and b in (4.1), can not be satisfied; since $p^* = 0$, for $\varphi' = 0$ from (4.4) $q\sin\varphi + q^*\cos\varphi = 0$ is obtained and so from (4.6), (4.7) $H' = 0$ again.

It is seen that in case of helicoids, the conditions $H = \text{const.}$ and $\varphi = \text{const.}$ are equivalent to each other. From (4.6), (4.7) and (4.4) ($p^* = 0$),

$$\begin{aligned} &H_1 = J(q^* + q \operatorname{tg} \varphi), \quad H_2 = -J(q^* \operatorname{cotg} \varphi + q), \quad J_1 = J(q \operatorname{tg} \varphi - q^*), \\ &J_2 = J(q^* \operatorname{cotg} \varphi - q), \quad \varphi_1 = \sin \varphi \, (q \sin \varphi + q^* \cos \varphi), \\ &\varphi_2 = -\cos \varphi \, (q \sin \varphi + q^* \cos \varphi) \end{aligned} \tag{4.8}$$

are obtained.

By the aid of the integrability condition from the first four equations of (4.8), we get

$$\begin{aligned} &(q_2 \operatorname{tg} \varphi + q_1) + (q_2^* + q_1^* \operatorname{cotg} \varphi) + q \left(\frac{\varphi_2}{\cos^2 \varphi} + q \operatorname{tg} \varphi + q^* \right) - \\ &\qquad - q^* \left(\frac{\varphi_1}{\sin^2 \varphi} - q^* \operatorname{cotg} \varphi - q \right) = 0, \\ &(q_2 \operatorname{tg} \varphi + q_1) - (q_1^* + q_2^* \operatorname{cotg} \varphi) + q \left(\frac{\varphi_2}{\cos^2 \varphi} + q \operatorname{tg} \varphi + q^* \right) + \\ &\qquad + q^* \left(\frac{\varphi_1}{\sin^2 \varphi} - q^* \operatorname{cotg} \varphi - q \right) = 0. \end{aligned} \tag{4.9}$$

By substituting the values in fifth and sixth equations of (4.8) of φ_1 and φ_2 in the equations (4.9), we obtain

$$q_1 \cos \varphi + q_2 \sin \varphi = q_1^* \cos \varphi + q_2^* \sin \varphi = 0 \tag{4.10}$$

which means that $q = q(T)$ and $q^* = q^*(T)$.

Conversely, if $H = H(T)$, $J = J(T)$, $q = q(T)$ and $q^* = q^*(T)$, then the first four equations of (4.8), and (4.10) can be written; so (4.9), (4.10) remain valid. Hence, the fifth and the sixth equations of (4.8) are obtained for φ_1 and φ_2. This means that, according to the fifth equation of (4.4) and the third equation of (4.6), $\varphi = \varphi(T)$ and $p^* = 0$. Finally, in order to show that the surface is a helicoid, we must prove that $p = p(T)$, i.e., $P_{1^*} = 0$. Clearly, with the values of φ_1 and φ_2 in (4.8), from (4.4) $p = q \cos \varphi - q^* \sin \varphi$ is found. So $p = p(T)$ and the theorem is proved.

4.3. Now let us apply the above criterium to the surfaces to be determined by the functions $a(x)$ and $b(y)$ in (4.1). By means of (4.1), the expressions (3.2) can be written as

$$E^0 = \frac{e^{x-y}}{(a_0 e^{2x} + c_0) \, | J |}, \quad G^0 = \frac{e^t}{(b_0 - c_0 e^{2y}) \, | J |}, \quad (F^0 = 0). \tag{4.11}$$

According to (1.1)

$$q = \frac{1}{2g^0} (\ln E^0)_y, \quad q^* = \frac{1}{2e^0} (\ln G^0)_x.$$

Thus, from (4.11), we have

$$q = -\frac{1}{2}\left(1+\frac{J'}{J}\right) c^{-\frac{t}{2}} \sqrt{|J|(b_0 - c_0 e^{2y})}, \quad q^* = \frac{1}{2}\left(1-\frac{J'}{J}\right) e^{\frac{t}{2}} \sqrt{|J|(a + c_0 e^{-2x})}.$$

It is seen that a necessary and sufficient condition to be $q = q(t)$, $q^* = q^*(t)$, that is, for the surface to be a helicoid, is that $c_0 = 0$. Hence the case $c_0 \neq 0$, corresponds to a class of the isothermic W-surfaces which are not helicoids.

Since the equation (4.2), from which the function $H(t)$ is obtained, does not have the constant c_0, this equation is valid for both the isothermic helicoids ($c_0 = 0$) and the new isothermic W-surfaces ($c_0 \neq 0$).

Accordingly, if the solution $H(t)$ of the equation (4.2) is found for the helicoids, by replacing the constants a_0 and b_0 in this solution by the constants a_0 and b_0 which are mentioned for the new surfaces, we directly obtain the function $H(t)$ of our new surfaces.

4.4. Finally, we will prove the following theorem:

Theorem. An isothermic W-surface which can not isometrically mapped to a surface of revolution is a surface of constant mean curvature.

Proof. We know that the isothermic helicoids, the surfaces of revolution, cones and cylinders can be isometrically mapped to the surface of revolution, and we also know that some of the surfaces of constant mean curvature can not be isometrically mapped to a surface of revolution. Therefore, we must show that the new surfaces, which are obtained for $c_0 \neq 0$ above, can be isometrically mapped to a surface of revolution.

As known, a necessary and sufficient condition for a surface to be isometrically mapped to a surface of revolution, is that $\Delta_1(K) = f(K)$, $\Delta_2(K) = g(K)$ [3, p. 92].

Since $K = H^2(t) - J^2(t) = K(t)$, $\Delta_1 K = K'^2 \Delta_1 t$ and $\Delta_2 K = K' \Delta_2 t + K'' \Delta_1 t$ can be written; so it is sufficient to show that $\Delta_1 t = m(t)$, $\Delta_2 t = n(t)$.

From the definitions of Δ_1 and Δ_2 [3, pp. 15, 16], we have

$$\Delta_1 t = \frac{1}{E^0} + \frac{1}{G^0}, \quad \Delta_2 t = \frac{i}{2E^0}\left(\ln \frac{G^0}{E^0}\right)_x + \frac{1}{2G^0}\left(\ln \frac{E^0}{G^0}\right)_y, \quad (t = x + y).$$

By substituting the values in (4.11) of E^0 and G^0 in the last expressions, we obtain

$$\Delta_1 t = e^{y-x} \mid J \mid (a_0 e^{2x} + b_0 e^{-2y}) = \mid J \mid (a_0 e^t + b_0 e^{-t}) = m(t);$$

$$\Delta_2 t = \frac{a'}{2} \cdot \frac{1}{a E^0} + \frac{b'}{2} \cdot \frac{1}{b G^0} = \mid J \mid e^{y-x} (a_0 e^{2x} - b_0 e^{-2y}) =$$

$$= \mid J \mid (a_0 e^t - b_0 e^{-t}) = n(t).$$

Therefore, the theorem is proved.

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