ON IRRESOLUTE MAPS AND SMALL IMAGES

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Summary : We demonstrate the behavioral similarity between the continuous and closed and irresolute and presemiclosed maps by proving the following: $f: (X, \tau) \to (Y, \sigma)$ is an irresolute and presemiclosed map if and only if $f(sCl(A)) = sCl(f(A))$ for all $A \subset X$. We also prove that if $f: (X, \tau) \to (Y, \sigma)$ is s-irreducible, irresolute and presemiclosed surjection, then:

(1) $f^{\#}(U) \in SO(Y, \sigma)$, $f^{\#}(U) \neq \phi$ and $f^{-1}(f^{\#}(U))$ is s-dense in U;

(2) $f(sCl(U)) = sCl(f^{\#}(U))$ for any non-empty set $U \in \tau$.

"IRRESOLUTE" TASVİRLER VE KÜÇÜK GÖRÜNTÜLER HAKKINDA

Özet : Bu çalışmada, bir $f: (X, \tau) \to (Y, \sigma)$ tasvirinin "irresolute" ve "presemiclosed" olabilmesi için gerek ve yeter koşulun bütün $A \subset X$ ler için $f(sCl(A)) = sCl(f(A))$ olduğu ispat edilerek sürekli, kapalı, "irresolute" ve "presemiclosed" tasvirler arasındaki benzerlik gösterilmekte ve bundan başka şu ispat edilmektedir: $f: (X, \tau) \to (Y, \sigma)$ s-indirgenemez "irresolute" ve "presemiclosed" bir sürjeksiyon ise

(1) $f^{\#}(U) \in SO(Y, \sigma)$, $f^{\#}(U) \neq \phi$ dır ve $f^{-1}(f^{\#}(U))$, U da s-yoğundur,

(2) Boş olmayan herhangi bir $U \in \tau$ için $f(sCl(U)) = sCl(f^{\#}(U))$ dur.

1. INTRODUCTION

Let $f: (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) to a topological space (Y, σ). If A is a subset of a topological space, then the abbreviations $Cl(A)$, $Int(A)$ and $sCl(A)$ shall denote the closure of A, the interior of A, and the semiclosure of A respectively. It is known that :

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(a) f is continuous and closed if and only if $f(Cl(A)) = Cl(f(A))$ for $A \subset X$,

and

(b) If f is continuous, irreducible, closed surjection, then for any non-empty set $U \in \tau$, $f(Cl(U)) = Cl(f^{\#}(U))$, where $f^{\#}(U)$ is the small image of U under f.

Keeping in view the striking similarities between the behaviors of continuous maps and irresolute maps in relation to open sets and semiopen sets respectively, one is naturally guided to the following pertinent questions analogous to (a) and (b) :

(c) Is f irresolute and presemiclosed if and only if $f(s(ClA)) = sCl(f(A))$ for every $A \subset X$?

(d) Does f irresolute. s-irreducible, presemiclosed surjection imply that $f(sCl(U)) = sCl(f^{\#}(U))$ for every semiopen set U in (X, σ)?

Quite predictably, the answers to both these questions are in affirmative (if the topology is suitably restricted).

2. PRELIMINARIES

A subset A of a topological space (X, τ) is called:

(1) regular open if $A = Int(Cl(A))$;

(2) regular closed if $A = Cl(Int(A))$;

(3) semiopen if $A \subset Cl(Int(A))$ [6] ;

(4) semiclosed if it is the complement of a semiopen set;

(5) preopen if $A \subset Int(Cl(A))$ [7];

(6) (X, τ) is called a D-space if every non-empty open set U is dense in X [7].

The intersection of all semiclosed sets containing A is called the semiclosure of A and is denoted by $sCl(A)$ [1]. Moreover, A is called s-dense in X if $sCl(A) = X$. In the following [8], the classes of regular open, regular closed, semiopen, semiclosed, and preopen sets shall be designated by $RO(X, \tau)$, $RC(X, \tau)$, $SO(X, \tau)$, $SC(X, \tau)$ and $PO(X, \tau)$ respectively.

A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is called :

(I) irresolute if for each $V \in SO(Y, \sigma)$, $f^{-1}(V) \in SO(X, \tau)$ [2] ;

Equivalently, $f: (X, \tau) \rightarrow (Y, \sigma)$ is irresolute if and only if $f(sCl(A)) \subset$ $\subset sCl(f(A))$ for all $A \subset X$.

(II) presemiclosed if for each $A \in SC(X, \tau)$, $f(A) \in SC(Y, \sigma)$ [10] ;

(III) s-irreducible if for any $F \in SC(X, \tau)$, $f(F) = Y$ implies that $F = X$.

Moreover, for any $A \subset X$, if f is surjection,

$$f^{+}(A) = \{y \in Y \mid \{f^{-1}(y)\} \subset A\}$$

is called the small image of A under f.

To make conspicuous the intimate relations between the foregoing concepts, we cite the following:

Lemma 2.1 [5, 8]. If A is a subset of a topological space (X, τ), then $A \in PO(X, \tau)$ if and only if $Int(Cl(A)) = sCl(A)$.

The proof directly follows from the fact that for every subset A of X, $Int(Cl(A)) \subset sCl(A)$.

3. Irresolute maps and semiclosures

Following equivalent formulation of the definition of an irresolute map can easily be proved.

Theorem 3.1. A map $f\colon (X, \tau) \to (Y, \sigma)$ is irresolute if and only if for each $x \in X$ and for every $V \in SO(Y, \sigma)$ containing $f(x)$, there exists $U \in SO(X, \tau)$ containing x such that $f(U) \subset V$.

Theorem 3.2. A map $f\colon (X, \tau) \to (Y, \sigma)$ is irresolute and presemiclosed if and only if for every $A \subset X$, $f(sCl(A)) = sCl(f(A))$.

Proof. Suppose that f is both irresolute and presemiclosed. Then irresoluteness of f implies that for all $A \subset X$,

$$f(sCl(A)) \subset sCl(f(A)). \tag{1}$$

Since f is presemiclosed, $f(sCl(A)) \in SC(Y, \sigma)$. Now $f(A) \subset f(sCl(A))$ gives

$$sCl(f(A)) \subset sCl(f(sCl(A))) = f(sCl(A)). \tag{2}$$

Combining (1) and (2), we get the required conclusion.

Conversely, we assume that for all $A \subset X$, $f(sCl(A)) = sCl(f(A))$. Then obviously f is irresolute. If $A \in SC(X, \tau)$, then $A = sCl(A)$, which in turn, gives that $f(A) = f(sCl(A)) = sCl(f(A))$, and hence $f(A) \in SC(Y, \sigma)$. Therefore f is presemiclosed.

Remark 3.1. It is not difficult to note that an alternative proof of our Theorem 3.2 may be obtained by using [4, Theorem 3.3].

Corollary 3.1. If f g : $(X, \tau) \to (Y, \sigma)$ are both irresolute, presemiclosed, and $A = \{x \in X \mid f(x) = g(x)\}$ is s-dense in X, then $f(X) = g(X)$.

4. Irresolute maps and small images

As a ready reference, we summarize some of the well-known results necessary for the proof of the ensuing theorem in the form of the following lemma:

Lemma 4.1. Let $f: (X, \tau) \to (Y, \sigma)$ be continuous, irreducible, closed surjection. Then :

(a) $Cl(f(A)) = f(Cl(A))$ for $A \subset X$;

(b) $f^{\#}(U) \in \sigma$ and $f(Cl(U)) = Cl(f^{\#}(U))$ for $U \in \tau$;

(c) $Int(Cl(A)) = X \setminus Cl(X \setminus Cl(A))$ for $A \subset X$;

(d) $X \setminus Cl(A) = Int(X \setminus A)$ for $A \subset X$;

(e) $f(X \setminus A) = Y \setminus f^{\#}(A)$ and $f(A) = Y - f^{\#}(X \setminus A)$ for $A \subset X$.

Remark 4.1. Combining (c) and (d), we get $Int(Cl(A)) = X \setminus Cl(Int(X \setminus A))$, which shows that $A \in RO(X, \tau)$ if and only if $X \setminus A \in RC(X, \tau)$.

Theorem 4.1. Let $f: (X, \tau) \to (Y, \sigma)$ be continuous, irreducible, closed surjection and let $A \in RO(X, \tau)$. Then $f(A) \in RO(X, \tau)$ if and only if f is injective.

Proof. Using (c) of Lemma 4.1, we get

$$\begin{aligned} Int(Cl(f(A))) &= Y \setminus Cl(Y \setminus Cl(f(A))) \\ &= Y \setminus Cl(Y \setminus f(Cl(A))), \text{ as } f \text{ is closed,} \\ &= Y \setminus Cl(Y \setminus Cl(f^{\#}(A))) \\ &= Y \setminus Cl(Int(Y \setminus f^{\#}(A))) \\ &= Y \setminus Cl(Int(f(X \setminus A))) \\ &= Y \setminus f(X \setminus A), \text{ as } X \setminus A \in RC(X, \tau), \\ &= f^{\#}(A). \end{aligned}$$

Now $f^{\#}(A) = f(A)$ if and only if f is injective. Thus $Int(Cl(f(A))) = f(A)$ if and only if f is injective.

To prove our main result that the relationship between the continuous maps and open sets is quite similar to the relationship between the irresolute maps and semiopen sets, we require the following:

Lemma 4.2. Let (X, τ) be a D-space. Then

(1) $U, V \in SO(X, \tau)$ implies that $U \cap V \in SO(X, \tau)$;

(2) if $U, V \in SO(X, \tau)$, $V \subset U$ and V is s-dense in U, then $sCl(V) = sCl(U)$.

Proof. Definition of D-space gives

$$Cl(U \cap V) = Cl(U) \cap Cl(V),$$

which, via the definition of semiopen set, proves the desired result.

(2) Let $sCl_U(V)$ denote the semiclosure of V in U. Then

$$sCl_U(V) = U.$$

Let $x \in sCl(U)$ and let $W \in SO(X, \tau)$ contain x. Then $W \cap U \neq \phi$ and $W \cap U \in SO(X, \tau)$. Noting that $W \cap U \in SO(U, \tau_U)$, we have $W \cap U \cap V \neq \phi$, which yields $W \cap V \neq \phi$, and hence $x \in sCl(V)$. Thus $sCl(V) = sCl(U)$.

Theorem 4.2. If (X, τ) is a D-space and $f: (X, \tau) \to (Y, \sigma)$ is an s-irreducible, irresulote and presemiclosed surjective map, then the following hold for any non-empty set $U \in SO(X, \tau)$:

(1) $f^{\#}(U) \in SO(Y, \sigma)$, $f^{\#}(U) \neq \phi$ and $f^{-1}(f^{\#}(U))$ is s-dense in U;

(2) $f(sCl(U)) = sCl(f^{\#}(U))$.

Proof. (1) For any subset A of (X, τ), we have $f^{\#}(A) = Y \setminus f(X \setminus A)$. Replacing A by a non-empty semiopen set U, we get

$$f^{\#}(U) = Y \setminus f(X \setminus U).$$

Since f is presemiclosed, $f^{\#}(U) \in SO(Y, \sigma)$. Further, s-irreducibility of f implies that $f^{\#}(U) \neq \phi$. Obvious set-theoretic considerations lead to $f^{-1}(f^{\#}(U)) \subset U$.

Let $W \in SO(X, \tau)$ be such that $W \subset U$. Then W is semiopen in U [10]. Therefore, $f^{\#}(W) \subset f^{\#}(U)$ implies that $W \cap f^{-1}(f^{\#}(U)) \neq \phi$. Thus every semiopen set W in U meets $f^{-1}(f^{\#}(U))$ and consequently

$$sCl_U(f^{-1}(f^{\#}(U))) = U.$$

By Lemma 4.2, we have

$$sCl(f^{-1}(f^{\#}(U))) = sCl(U).$$

(2) Now

$$\begin{aligned} f(sCl(U)) &= f(sCl(f^{-1}(f^{\#}(U)))) \\ &= sCl(f(f^{-1}(f^{\#}(U)))), \text{ (Theorem 3.2)} \\ &= sCl(ff^{-1}(f^{\#}(U))), \text{ as } f \text{ is surjective} \\ &= sCl(f^{\#}(U)). \end{aligned}$$

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