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THE STRUCTURE OF THE GROUPS ALL WHOSE SUBGROUPS ARE NORMAL

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Summary : In this paper the following theorem is proved: Let G be a finite group all whose subgroups are normal. Then G is nilpotent, the Sylow p-subgroups for odd prime p are abelian and the Sylow 2-subgroups are metabelian.

BÜTÜN ALT GRUPLARI NORMAL OLAN GRUPLARIN YAPISI

Özet : Bu çalışmada şu teorem ispat edilmektedir: G, bütün alt grupları normal olan sonlu bir grup olsun. Bu takdirde G nilpotenttir, tek p asal sayıları için p-Sylow alt grupları abelyendir ve 2-Sylow alt grupları metabelyendir.

In [1] Fujikawa proved the following :

Theorem. Let G be a finite group. Then $n(\chi)=\chi(1)$ for all $\chi \in Irr(G)$ iff every subgroup of G is normal ($n(\chi)$ the permutation index (see [1])).

The next statement is a similar result about Schur index and can be found in [3], pg. 173.

Proposition. Let G be a finite group. Suppose that $m_F(\chi) = \chi(1)$ for all $\chi \in Irr(G)$ with $F \subseteq \mathbf{C}$. Then every subgroup of G is normal.

These results lead us to study the structure of the finite groups all whose subgroups are normal.

The abelian group and the quaternion group are examples of such groups. The class of these groups was studied by Dedekind (see [2]). In this paper we shall study the structure of these groups using the character theory of groups and we shall derive some interesting consequences of this.

We shall denote by $\mathscr{A}$ the class of the finite groups all whose subgroups are normal. Notation and terminology are standard (cf. [3] and [4] for example).

Proposition 1. Let $G \in \mathscr{A}$ and $H < G$. Then:

i) $H \in \mathscr{A}$.

ii) $G/H \in \mathscr{A}$.

iii) $\varkappa \in Z(G)$ for every involution $\varkappa$ of G.

Proof. i) and iii) are obvious and ii) follows by the isomorphism theorem for groups (see [4], pg. 50).

Theorem 2. Let $G \in \mathscr{A}$. Then:

i) G is nilpotent.

ii) The Sylow p-groups of G for odd primes p are abelian.

iii) Let $S \in Syl_2(G)$ nonabelian. Then S is a metabelian group, all its irreducible characters have degrees 1 or 2 and there exist $H<S$ abelian of index 2 or $[G : Z(G)]=8$. Besides $G/ker(\chi)$ is a generalized quaternion group for every nonlinear $\chi \in Irr(G)$.

Proof. Step 1. G is nilpotent since all Sylow subgroups of G are normal subgroups.

Remark. Since G is nilpotent, by Prop. 1 it is sufficient to study the structure of the p-groups of $\mathscr{A}$.

Step 2. The p-groups of $\mathscr{A}$ with p an odd prime are abelian.

Let G be a minimal counter-example and $\chi \in Irr(G)$ non-linear. Then χ is faithful and $Z(G)$ is cyclic. Let M be a maximal abelian subgroup of G. We shall prove that M is cyclic.

Since M is normal in G, by Clifford's theorem (see [3]) $\chi_M = e \sum\limits_{g \in G-M} \tau^g$, where $\tau \in Irr(M)$. If τ is invariant in G, then $\chi_M = e\,\tau$ and τ is an irreducible faithful character of M, hence M is cyclic. If not, $\chi_M = \sum\limits_{g \in G-M} \tau^g$ and then $\chi = \tau^G$. Since χ is faithful and M is abelian it follows that τ is also faithful and hence M is cyclic. Then (see [2], pg. 311, Prop. 8.4) G is cyclic.

Step 3. Let G be a nonabelian 2-group of $\mathscr{A}$. Then G is a metabelian group and there exists an abelian subgroup H of G of index 2 or 4. Besides $G/ker(\chi)$ is a generalized quaternion group for every nonlinear $\chi \in Irr(G)$.

Let $\chi \in Irr(G)$ nonlinear. Let $H = G/ker(\chi)$. Then $H \in \mathscr{A}$ and has a nonlinear faithful character. Hence $Z(H)$ is cyclic and by Prop. 1 H contains only one involution. Therefore H is a generalized quaternion group. Since the generalized quaternion groups have irreducible characters of degrees at most 2, it follows that the irreducible characters of G have degrees also at most 2. Let $K < G$ maximal such that G/K is nonabelian and let $P < G$ such that $Z(G/K) = P/K$. Let $L = G/K$. Then L' is the unique minimal normal subgroup of L. Since $Z(L)$ is not trivial, $L' < Z(L)$ and $|L'| = 2$. Then every nonlinear $\chi \in Irr(L)$ is faithful and $\chi(1)^2 = |L : Z(L)|$ (see [3], pg. 28). Let $x, y \in L$. Since $[x, y] \in Z(L)$ and $|L'| = 2$, we have $[x^2, y] = [x, y]^2 = 1$. Thus $x^2 \in Z(L)$ for all $x \in L$. Therefore $Z(L)$ is cyclic and $L/Z(L)$ is an elementary abelian group of order 4. Let now $\alpha \in Irr(L)$ with $\alpha(1) = 2$ and $\tau \in Irr(P)$. Since $P < Z(\alpha)$, by Clifford's theorem $\tau(1) = 1$ and P is abelian.

Step 4. Let $G \in \mathscr{A}$ be a non-abelian 2-group. Then one of the following holds:

i) There exists $H < G$ abelian of index 2.

ii) The index of $Z(G)$ in G is 8.

Assume i) is false. By the previous step there exists $K < G$ abelian of index 4. Then K acts by conjugation on $G \setminus K$ and has an orbit of size 1 or 2. Since G does not have abelian subgroups of index 2, $A = C_G(K)$, therefore there exists $x \in G : K$ in an orbit of size 2. Let $H = \langle K, x \rangle$. Then $Z(H) = C_K(x)$ has index 2 in K and index 8 in G. We shall prove that $Z(G) = Z(H)$.

Let $L = Z(H)$ and $\chi \in Irr(G)$. If χ_H is irreducible, then $L < Z(\chi)$ and $[G : L] < ker(\chi)$. If not, then all irreducible constituents of χ_H are linear and $H' < ker(\chi)$. Suppose $[G : L]$ is non-trivial. Then $H' \cap [G : L] = 1$. Since $H'\,[G : L]$ is the direct product of two non-trivial groups it has an irreducible character τ such that H' is not included in $ker(\tau)$ and $[G : L] < ker(\tau)$. Let χ be an irreducible constituent of τ^G. Then H' and $[G : L]$ are not included in $ker(\chi)$, contradiction. Hence $[G : L] = 1$.

Corollary 3. Let G be a finite group. Then the following are equivalent:

i) $n(\chi) = \chi(1)$ for all $\chi \in Irr(G)$.

ii) $m_F(\chi) = \chi(1)$ for all $\chi \in Irr(G)$ with $F \subseteq \mathbb{C}$.

iii) $G \in \mathscr{A}$.

Proof. By Theorem 2 it is easy to see that if $G \in \mathscr{A}$, then $m_F(\chi) = \chi(1)$ for all $\chi \in Irr(G)$ with $F \subseteq \mathbb{C}$ so we have the converse to the Isaacs proposition quoted in introduction.

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