BIPARTITE GROUPS

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Summary : In this paper we study bipartite groups and bipartite presentations. Further, we show that if G is bipartite group, then G is closed under the subgroups of G. The result of this paper is useful to recognize what each defining relator look like.

"BIPARTITE" GRUPLAR

Özet : Bu çalışmada "bipartite" gruplar ve "bipartite" gösterilişler incelenmekte ve G nin "bipartite" grup olması durumunda G nin, alt grupları altında kapalı olduğu gösterilmektedir.

1. Introduction. Given a group presentation (or more generally a 2-complex $\mathscr{K}$) one can associate with it a 1-complex, called the star complex. The star complex $\mathscr{K}^{st}$ has been proved to be useful in several contexts (see [1], [3] and [5]). A group G is called a bipartite group if G is a fundamental group of a 2-complex $\mathscr{K}$ with a bipartite complex $\mathscr{K}^{st}$. In this paper, we show that if G is bipartite group, then G is closed under the subgroups of G. Further, we consider a bipartite presentation (a presentation is a 2-complex with a single vertex). The result of this paper is useful to recognize what each defining relator look like. Our methods involve a rather geometric version of combinatorial group theory.

2. Preliminaries. A 1-complex χ consists of two disjoint sets V (vertices), E (edges) together with three functions $\iota : E \to V$, $\tau : E \to V$, ${}^{-1} : E \to E$ satisfying $\iota(e^{-1}) = \tau(e)$, $(e^{-1})^{-1} = e$, $e^{-1} \neq e$ for all $e \in E$. A non-empty path α in χ is a sequence $e_1 e_2 \dots e_n$ $(n \geq 1)$ of edges with $\tau(e_i) = \iota(e_{i+1})$ $(1 \leq i \leq n)$. The inverse path α^{-1} of α is the path $e_n^{-1} \dots e_2^{-1} e_1^{-1}$. A path α is said to be closed if $\iota(\alpha) = \tau(\alpha)$. We say that a path α is reduced if $e_i \neq e_{i+1}^{-1}$ for all $i = 1, 2, \dots, n-1$. Moreover if α is closed, then we say that α is cyclically reduced if all its cyclic permutations are reduced. With each vertex v in χ, we associate the empty path 1_v.

This path has no edges.

A 1-complex χ is said to be connected if given any two vertices u, v in χ, there is a path γ such that $\iota(\gamma) = u$, and $\tau(\gamma) = v$. We remark that 1-complexes are called graphs by combinatorial group theorists (see [6]). In this paper a graph is a pair consisting of a set V of vertices and a subset E of $V \times V$. An element of E is called an edge. If $e = (v_1, v_2) \in E$, then e is the edge which joins the two vertices v_1 and v_2. For the general theory of graph theory, we refer the reader to [2].

Now, given any 1-complex χ with vertex set V, we associate with it a graph $\Gamma(\chi)$ in the following way: the vertex set of $\Gamma(\chi)$ is V, and $\{u, v\}$ is an edge of $\Gamma(\chi)$ if and only if there is an edge e of χ such that $\iota(e) = u$ and $\tau(e) = v$.

A 2-complex $\mathscr{K}$ is an object $< \chi ; p_\lambda (\lambda \in \Lambda) >$, where χ is a 1-complex and each p_λ is a closed path in χ, called the defining path. The elements of Λ are called indices. If $\mathscr{K}$ is a 2-complex, then we define $\mathscr{R}(\mathscr{K})$ as the set of all cyclic permutations of a non-empty defining paths with the inverses. Let $\mathscr{K}$ be a 2-complex. Then an equivalence relation $\sim_{\mathscr{K}}$ (or simply $\sim$) on paths in $\mathscr{K}$ as follows: An elementary reduction of a path α is a transformation of α to $\alpha_1 \alpha_2$ if α has one of the forms $\alpha_1 \gamma\gamma^{-1} \alpha_2$, $\alpha_1 p \alpha_2$, where γ is any path and $p \in \mathscr{R}(\mathscr{K})$. If α and $\alpha'$are two paths then we define $\alpha \sim_{\mathscr{K}} \alpha'$ if and only if there is a sequence of paths $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_m = \alpha'$, where for $i = 1, 2, \ldots, m-1$ one of α_i, α_{i+1} is obtained from the other by an elementary reduction. The $\sim$-equivalence class containing α is denoted by $[\alpha]_{\mathscr{K}}$ (or simply $[\alpha]$). A path which is $\sim$-equivalent to an empty path is called contractible. If α and β are two paths such that $\alpha\beta$ is defined, then we define $[\alpha][\beta] = [\alpha\beta]$. Let $\mathscr{K}$ be a 2-complex and let v be a vertex of $\mathscr{K}$. Then the fundamental group $\pi_1(\mathscr{K}, v)$ of $\mathscr{K}$ at v has the underlying set $\{[\alpha] : \alpha$ a closed path with $\iota(\alpha) = v\}$, where the binary operation is the product defined above, and $[\alpha]^{-1} = [\alpha^{-1}]$. If $\mathscr{K}$ is connected, then the fundamental group $\pi_1(\mathscr{K}, v)$ is independent of the choice of the base vertex v of $\mathscr{K}$. Therefore, we can refer to the fundamental group of $\mathscr{K}$.

Now, let $\mathscr{K}$ and $\mathscr{L}$ be 2-complexes. Then a mapping $\phi : \mathscr{K} \to \mathscr{L}$ is a function taking vertices to vertices, paths to paths satisfying $\phi(1_v) = 1_{\phi(v)}$, for a vertex v of $\mathscr{K}$, $\phi(\alpha_1 \alpha_2) = \phi(\alpha_1)\phi(\alpha_2)$, α_1, α_2 are paths in $\mathscr{K}$ and $\alpha_1 \alpha_2$ is defined, $\phi(\alpha^{-1}) \sim_{\mathscr{L}} \phi(\alpha)^{-1}$; $\phi(p)$ is contractible in $\mathscr{L}$ for each defining path ρ of $\mathscr{K}$. We say that ϕ is rigid if it maps edges to edges, and satisfies $\phi(\alpha^{-1}) = \phi(\alpha)^{-1}$ for all paths α in $\mathscr{K}$. If $\phi : \mathscr{K} \to \mathscr{L}$ is a mapping, then there exists an induced homomorphism $\phi_* : \pi_1(\mathscr{K}, v) \to \pi_2(\mathscr{L}, \phi(v))$ defined by $\phi_*([\alpha]_{\mathscr{K}}) = [\phi(\alpha)]_{\mathscr{L}}$, where $[\alpha]_{\mathscr{K}} \in \pi_1(\mathscr{K}, v)$. A mapping $\phi : \mathscr{K} \to \mathscr{L}$ is said to be locally bijective if ϕ is rigid and maps $\mathrm{Star}(v) = \{e : e \in E(\mathscr{K}), \iota(e) = v\}$ bijectively onto $\mathrm{Star}(\phi(v))$, $\phi^{-1}(\mathscr{R}(\mathscr{L})) = \mathscr{R}(\mathscr{K})$. If, in addition, $\mathscr{K}$ and $\mathscr{L}$ are connected, then ϕ is called covering (see [1], [5]). Let $\mathscr{K}$ be a connected 2-complex, ν a vertex of $\mathcal{K}$, and let H be a subgroup of $\pi_1(\mathcal{K}, \nu)$. Then there is a covering $\phi_H : \rightarrow \mathcal{K}$ and a vertex ν_H of $\mathcal{K}_H$ such that ϕ_{H*} maps $\pi_1(\mathcal{K}_H, \nu_H)$ isomorphically onto H, where ϕ_H is the covering corresponding to H.

Let $\mathcal{K}$ be a 2-complex. Then we can associate with $\mathcal{K}$ a 1-complex $\mathcal{K}^{st}$ called the star complex of $\mathcal{K}$ by taking vertices $E(\mathcal{K})$, and edges $\mathcal{R}(\mathcal{K})$. If γ is an edge of $\mathcal{K}^{st}$, then we can define the inverse edge to be the inverse path γ^{-1}. In order to define the initial and terminal points of γ, we use the notation $\iota^{st}(\gamma)$, $\tau^{st}(\gamma)$. We define $\iota^{st}(\gamma)$ to be the first edge of γ and $\tau^{st}(\gamma)$ to be the inverse of the last edge of γ. We note that $\mathcal{K}^{st}$ is a 1-complex provided that no element of $\mathcal{R}(\mathcal{K})$ is equal to its inverse.

A 2-complex $\mathcal{K}$ satisfies $T(4)$ — condition if and only if $\mathrm{T}(\mathcal{K}^{st})$ has no triangles (see [3]).

3. Subgroups of bipartite groups. A 1-complex χ with a vertex set V is called a bipartite complex if V is disjoint union of two non-empty sets V_1, V_2 such that if e is an edge of χ, then one of $\iota(e)$, $\tau(e)$ belongs to V_1 and the other to V_2, that is, χ is a bipartite complex if $\Gamma(\chi)$ is a bipartite graph. We point out that a bipartite graph has no reduced closed path of odd length. In particular a bipartite graph has no triangles. It follows that a bipartite complex is a $T(4)$— complex (see [3]).

Theorem 1. Every subgroup of a bipartite group is a bipartite group.

In order to prove Theorem 1, we prove the following lemma:

Lemma 1. Let $\phi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be a locally bijective mapping of 1-complexes $\tilde{\mathcal{C}}$ and $\mathcal{C}$. If $\mathcal{C}$ is a bipartite complex, then so $\tilde{\mathcal{C}}$.

Proof. Define a vertex $\tilde{\nu} \in \tilde{\mathcal{C}}$ to be an element in $\tilde{V}_1$ or $\tilde{V}_2$ if $\phi(\tilde{\nu}) = \nu$ is an element in V_1 or V_2. Then we have $\tilde{V} = \tilde{V}_1 \dot{\cup} \tilde{V}_2$. Now, let $\tilde{e}$ be an edge of $\tilde{\mathcal{C}}$. Then by the definition of $\phi(\tilde{e}) = e$ is an edge of $\mathcal{C}$ has one of $\iota(e)$, $\tau(e)$ belonging to V_1 and the other to V_2. But ϕ is locally bijective. Then one of $\iota(e)$, $\tau(e)$ belongs to $\tilde{V}_1$ and the other to $\tilde{V}_2$. Therefore, $\tilde{\mathcal{C}}$ is a bipartite complex.

Proof of Theorem 1. Let $\mathcal{K}$ be a 2-complex with a bipartite complex $\mathcal{K}^{st}$. Then $G \cong \pi_1(\mathcal{K})$ is a bipartite group. Let $H \leq \pi_1(\mathcal{K})$, and let $\mathcal{K}_H$ be a covering of $\mathcal{K}$, where $\pi_1(\mathcal{K}_H) \cong H$. Then we want to prove that $\Gamma(\mathcal{K}_H^{st})$ as $\Gamma(\mathcal{K}^{st})$, that is $\mathcal{K}_H^{st}$ is a bipartite complex if $\mathcal{K}^{st}$ is a bipartite complex. Let $\eta : \mathcal{K}_H \rightarrow \mathcal{K}$ be locally bijective of 2-complexes. By the fact that $\eta^{st} : \mathcal{K}_H^{st} \rightarrow \mathcal{K}^{st}$ is locally bijective mapping of 1-complexes (see [5]), and by Lemma 1, we have $\Gamma(\mathscr{K}_H^{st})$ as $\Gamma(\mathscr{K}^{st})$.

4. Bipartite presentations. In order to state our result we recall the following definition (see [5]). Let $\mathbf{x}$ be an alphabet. A word w on $\mathbf{x}$ is said to be positive (resp. negative) if only positive (resp. negative) powers of elements of $\mathbf{x}$ appear in the word. Let $\mathbf{u}$ and v be two disjoint sets of non-empty words on $\mathbf{x}$. A word $t_1 t_2 \ldots t_s (s \geq 1,\ t_i \in \mathbf{u} \cup \mathrm{v}$, for $i = 1, 2, \ldots s)$ is called an alternating $(\mathbf{u}, \mathrm{v})$ - word if the following holds: for $1 \leq i \leq s$, if $t_i \in \mathbf{u}$, then $t_{i+1} \in \mathrm{v}$. A word is called a cyclically alternating $(\mathbf{u}, \mathrm{v})$ - word with the additional property that t_1 and t_s do not belong to the same set $\mathbf{u}$ or v.

Consider a bipartite presentation having n generators $\mathbf{x}$. Let $\mathbf{x} = \mathbf{x}_1 \dot{\cup} \mathbf{x}_2 \dot{\cup} \mathbf{x}_3$, where the numbers of generators $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are k, l, m respectively such that $k + l + m = n$.

Theorem 2. Let $\mathscr{P}$ be a bipartite presentation as described above. Then each defining relator is a positive word on $\mathbf{x}_1$ (up to replacing each generator by its inverse) or cyclically alternating $(\mathbf{x}_2^{\pm}, \mathbf{x}_3^{\pm})$ - word or cyclically alternating $(\mathbf{u}, \mathrm{v})$ - word, where $\mathbf{u}$ is a positive word on $\mathbf{x}_1$ and v is an alternating $(\mathbf{x}_2^{\pm}, \mathbf{x}_3^{\pm})$ - word.

Proof. The vertex set of $\Gamma(\mathscr{P}^{st})$ looks like :

$$\mathbf{x}_1 \left\{ \begin{array}{llll} x_1 & \bullet & \bullet & x_1^{-1} \\ x_2 & \bullet & \bullet & x_2^{-1} \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ x_k & \bullet & \bullet & x_k^{-1} \end{array} \right.$$

$$\mathbf{x}_1^{\pm 1} \left\{ \begin{array}{llll} x_{k+1} & \bullet & \bullet & x_{k+l+1} \\ x_{k+1}^{-1} & \bullet & \bullet & x_{k+l+1}^{-1} \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ x_{k+l} & \bullet & \bullet & x_{k+2l} \\ x_{k+l}^{-1} & \bullet & \bullet & x_{k+2l}^{-1} \\ & & \bullet & \\ & & \bullet & \\ & & \bullet & \\ & & \bullet & x_{k+l+m} \\ & & \bullet & x_{k+l+m}^{-1} \end{array} \right\} \mathbf{x}_3^{\pm 1}$$

Note that for a presentation $\mathscr{P} = < \mathbf{x} ; \mathrm{r} >$ that $\{x^{\varepsilon}, y^{\delta}\}$ $(x, y \in \mathbf{x}, |\varepsilon| = |\delta| = 1)$ is an edge if and only if $y^{-\delta} x^{\varepsilon}$ is a subword of some elements of r^* where r^* is the set of all cyclic permutations of elements of r and their inverses.

Case 1. If $\mathbf{x}_2 = \mathbf{x}_3 = \phi$, then each defining relator will be a positive word on $\mathbf{x}_1$ (up to replacing each generator by the inverse).

Case 2. If $\mathbf{x}_1 = \phi$, then each defining relator is cyclically alternating (u, v) - word where u is a positive word on $\mathbf{x}_1$ and v is an alternating $(\mathbf{x}_2^{\pm}, \mathbf{x}_3^{\pm})$ - word.

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