ON THE FUZZY CONDITIONAL SEQUENCE ENTROPY FUNCTION

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Summary : In this paper we recall some properties of fuzzy dynamical systems and non-fuzzy conditional sequence entropy function without going into details. After that we define the fuzzy conditional sequence entropy function. Finally, we prove some basic properties of this function.

FUZZY DİZİSEL KOŞULLU ENTROPİ FONKSİYONU HAKKINDA

Özet : Bu çalışmada ilk olarak detaylara girmeksizin, fuzzy dinamik sistemler ve fuzzy olmayan dizisel koşullu entropi fonksiyonu tanımlanmaktadır. Daha sonra fuzzy dizisel koşullu entropi fonksiyonu tanımlanmaktadır. Son olarak ta bu fonksiyonun bazı temel özelikleri ispat edilmektedir.

1. INTRODUCTION

Hulse and Zhang introduced the concepts of conditional sequence entropy function and investigated some basic properties of this function in (5) and (14).

The author has recently proved in (10) some properties of fuzzy conditional non-sequence entropy function and stated in (11) some results of the fuzzy conditional sequence information function.

It is the purpose of this work to define the fuzzy conditional sequence entropy function and to prove some important properties of this function.

2. CONDITIONAL SEQUENCE ENTROPY FUNCTION

Let (X, A, m, S) be a dynamical system, where (X, A, m) is a measure space and $S : X \to X$ is a measure-preserving transformation.

Here we follow mainly [1], [3], [5], [7] and [13].

2.1 Definition. A dynamical system (Y, B, m_0, S_0) is a factor of (X, A, m, S) if there exists a measure-preserving map $\varphi : X \to Y$ such that $\varphi \circ S = S \circ \varphi$. Equivalently, we say that (X, A, m, S) is an extension of (Y, B, m_0, S_0).

Key Words : Dynamical system, conditional entropy function, conditional sequence entropy function, fuzzy dynamical systems, fuzzy conditional sequence entropy function.

Let $P = \{A_1, \ldots, A_n\}$ be a finite measurable partition of X. The function

$$H_m(P/B) = \int \sum_{i=1}^{n} Z(E_m(1_{A_i}/B))\, dm$$

$$= \int \sum_{i=1}^{n} Z(m(A_i/B))\, dm$$

is called conditional entropy function of the partition P with respect to the σ-algebra B, where the function $Z = [0, \infty) \to \mathbf{R}$ defined by

$$Z(x) = \begin{cases} -x \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a non-negative continuous and strictly concave function. $E_m(1_{A_i}/B)$ denotes the conditional expectation of the characteristic function 1_{A_i} with respect to σ-algebra B (See [1]). We always understand 'log' to be natural logarithm.

2.2 Proposition. Let P and Q be two finite partitions of the dynamical system (X, A, m, S) and (Y, B, m_0, S_0) be a factor of (X, A, m, S). Then,

(i) $H_m(P/B) \geq 0$. Equality holds iff $P \subset B$

(ii) $H_m(P \cup Q/B) = H_m(P/B) + H_m(Q/P \cup B)$

(iii) If $P \subset Q$ then $H_m(P/B) \leq H_m(Q/B)$

(iv) $H_m(P \cup Q/B) \leq H_m(P/B) + H_m(Q/B)$

(v) If S is measure-preserving, then $H_m(SP/B) = H_m(P/B)$

(vi) $H_m(P/B) \leq H_m(P)$.

See [1] and [13].

2.3 Lemma. Let (a_n) be a sequence which is positive and subadditive. Then

$$\lim_{n\to\infty} \frac{1}{n} a_n \text{ exists and is equal to } \inf_{n} \frac{1}{n} a_n.$$

Proof. See Lemma II-1 of [9].

2.4 Theorem. Suppose that (Y, B, m_0, S_0) is a factor of dynamical system (X, A, m, S) and S is invertible. If $D = (t_n)_{n\geq 1}$ is a sequence of integers and P is a finite measurable partition of X, then

$$\lim_{n\to\infty} \sup \frac{1}{n} H\left(\bigcup_{i=1}^{n} S^{t_i} P/B\right) \text{ exists.}$$

Proof. The result follows from (i) and (iv) of Proposition 2.2 and Lemma 2.3.

2.5 Definition. The limit of $\frac{1}{n} H\Big(\bigcup_{i=1}^{n} S^{t_i} P/B\Big)$ is called the non-fuzzy conditional sequence entropy function of P with respect to σ-algebra B. Denote this limit by $h_D(S, P/B)$. The function $h_D(S/B) = \sup\{h_D(S, P/B)\}$ is called the non-fuzzy conditional sequence entropy function of dynamical system, where the supremum is taken over finite measurable partition of X with $H(P/B) < \infty$. For other properties of this function see [12].

3. FUZZY CONDITIONAL SEQUENCE ENTROPY FUNCTION

Here we follow mainly [2], [4], [5], [6], [8] and [9].

3.1 Definition. A fuzzy set is a pair (X, A) where X is a non-empty set and $A : X \to [0, 1]$ is the membership function. The family of all fuzzy subsets of X will be denoted by F. This family is called fuzzy class.

If $\Big(\bigcup_{i=1}^{j} A_i\Big) \cap A_{j+1} = \phi, j = 1, 2, \ldots, n-1$ the collection $A_1, \ldots, A_n$ of fuzzy sets is called disjoint. A collection $P = \{A_1, \ldots, A_n\}$ of disjoint fuzzy sets is a finite fuzzy partition of X iff

$$X = \bigcup_{i=1}^{n} A_i .$$

Let A and B be two fuzzy sets. Then we define the product and the difference of fuzzy sets A and B by

$$(A.B)(x) = A(x).B(x) \quad \forall x \in X$$

$$(A - B)(x) = \max(A(x) - B(x), 0) \quad \forall x \in X.$$

The complement of A is a fuzzy set $\bar{A}$ defined by $\bar{A}(x) = 1 - A(x)$ for every $x \in X$.

3.2 Definition. The fuzzy class F is called a fuzzy σ-algebra on X if it satisfies the following conditions:

(i) $X \in F$

(ii) If $A, B \in F$ then $A.B \in F$ and $A - B \in F$

(iii) If $A_n \in F (n = 1, 2, \ldots)$ then $\sup_n A_n \in F$.

The pair (X, F) is a fuzzy measurable space, the elements of F are fuzzy measurable sets. If Ihe mapping $\mu : (X, F) \to \mathbf{R}$ fulfills the following properties μ is called a fuzzy measure on the fuzzy measurable space (X, F).

(i) $\mu(A) \geq 0$ for every $A \in F$, $\mu(\phi) = 0$

(ii) If $(A_n)_{n\geq 1}$ is a disjoint sequence of fuzzy sets and $A_n \in F$ for every n, then we have

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

A triple (X, F, μ) is called a fuzzy measure space. If $\mu(X) = I$, this triple may be called a fuzzy probability space. In this case, the elements of F are called fuzzy events and μ is a fuzzy probability measure.

3.3 Definition. Let (X, F, μ) be a fuzzy probability space. A system $P = \{A_1, \ldots, A_n\}$ with $A_i \in F (i = 1, 2, \ldots, n)$ is called a complete system of fuzzy events iff P is a fuzzy partition of X. Let P and Q be two fuzzy complete systems. P and Q are independent iff

$$\mu(A.B) = \mu(A) . \mu(B).$$

$T : (X, F) \to (X, F)$ is called a σ-homomorphism if it satisfies the following conditions:

(i) $T(\bar{A}) = \overline{T(A)}$ for every $A \in F$

(ii) $T\left(\bigcup_n A_n\right) = \bigcup_n T(A_n)$ for any sequence $(A_n)_{n\geq 1} \subset F$.

The σ-homomorphism $T : (X, F) \to (X, F)$ is a measure-preserving if $\mu(TA) = \mu(A)$ for every $A \in F$. The quadruple (X, F, μ, T) is called a fuzzy dynamical system.

3.4 Theorem. Suppose that the fuzzy dynamical system (Y, F_1, μ_1, T_1) is a factor of fuzzy dynamical system (X, F, μ, T) and T is invertible. If $D = (t_n)_{n\geq 1}$ is a sequence of integers with $t_1 = 0$ and P is a fuzzy complete system, then

$$\lim_{n\to\infty} \sup \frac{1}{n} H\left(\bigcup_{i=1}^{n} T^{t_i} P/F_1\right) \text{ exists.}$$

Proof. Let $a_n = H\left(\bigcup_{i=1}^{n} T^{t_i} P/F_1\right)$. We shall prove that $a_n \geq 0$ and $a_{n+m} \leq \leq a_n + a_m$. It is clear that a_n is non-negative from the **Definition** 2.1 and (i) of Theorem 2.2.

Let $D = (t_i)$ be a sequence of integers with $t_1 = 0$. Put

$$t_j = \max_{n+1 \leq i \leq n+m} (t_i). \tag{3.1}$$

Then we write the following sequence:

$$(t_{n+k} - t_j)_{k=1}^{m} = (t_{n+1} - t_j, \ldots, t_{j-1} - t_j, t_j - t_j, t_{j+1} - t_j, \ldots, t_{n+m} - t_j)$$

with

$$\begin{aligned} t_{n+k} - t_j &= t_{k+1} \qquad 1 \leq k \leq j-n-1 \\ t_{n+l} - t_j &= t_l \qquad j-n+1 \leq l \leq m \\ t_j - t_j &= t_1 = 0. \end{aligned} \tag{3.2}$$

$$\begin{aligned} a_{n+m} &= H\Big(\bigcup_{i=1}^{n+m} T^{t_i} P / F_1\Big) \\ &= H\Big(\Big[\bigcup_{i=1}^{n} T^{t_i} P \cup \bigcup_{i=n+1}^{n+m} T^{t_i} P\Big] / F_1\Big) \text{ by (iv) of Proposition 2.2} \\ &\leq H\Big(\bigcup_{i=1}^{n} T^{t_i} P / F_1\Big) + H\Big(\bigcup_{i=n+1}^{n+m} T^{t_i} P / F_1\Big) \text{ by (v) of Proposition 2.2} \\ &= a_n + H\Big(\bigcup_{i=n+1}^{n+m} T^{t_i - t_j} P / F_1\Big) \\ &= a_n + H\Big(\Big[\bigcup_{k=1}^{j-n-1} T^{t_{k+1} - t_j} P \cup T^{t_j - t_j} P \cup \bigcup_{l=j-n+1}^{m} T^{t_{n+l} - t_j} P\Big] / F_1\Big) \text{ by (3.2)} \\ &= a_n + H\Big(\Big[\bigcup_{k=1}^{j-n-1} T^{t_{k+1}} P \cup T^{t_1} P \cup \bigcup_{l=j-n+1}^{m} T^{t_l} P\Big] / F_1\Big) \\ &= a_n + H\Big(\bigcup_{i=1}^{m} T^{t_i} P / F_1\Big) = a_n + a_m. \end{aligned}$$

We then apply Lemma 2.3. So,

$$\limsup_{n \to \infty} \frac{1}{n} H\Big(\bigcup_{i=1}^{n} T^{t_i} P / F_1\Big)$$

exists.

3.5 Definition. The limit of $\frac{1}{n} H\Big(\bigcup_{i=1}^{n} T^{t_i} P / F_1\Big)$ is called the fuzzy conditional sequence entropy function of P with respect to the fuzzy σ-algebra

F_1. Denote this limit by $h_D(T, P/F_1)$. The function $h_D(T/F_1) = \sup\{h_D(T, P/F_1)\}$ is called the fuzzy conditional sequence entropy function of fuzzy dynamical system, where the supremum is taken over fuzzy complete systems with $H(P/F_1) < \infty$.

3.6 Lemma. We consider (Y, F_1, μ_1, T_1), a factor of fuzzy dynamical system (X, F, μ, T). Let T be an invertible σ-homomorphism and $D = (t_n)_{n \geq 1}$ be a sequence of integers. If P is a fuzzy complete system of X, then

$$H\Big(\bigcup_{i=1}^{n} T^{t_i} P/F_1\Big) = H(P/F_1) + \sum_{i=2}^{n} H\Big(T^{t_i} P/\bigcup_{j=1}^{i-1} T^{t_j} P \cup F_1\Big).$$

In particular

$$h_D(T_1, P/F_1) = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=2}^{n} H\Big(T^{t_i} P/\bigcup_{j=1}^{i-1} T^{t_j} F \cup F_1\Big).$$

Proof. We show it by induction. This is true for $n = 2$. In fact

$$H\Big(\bigcup_{i=1}^{2} T^{t_i} P/F_1\Big) = H([T^{t_1} P \cup T^{t_2} P]/F_1) \text{ by (ii) of Proposition 2.2}$$

$$= H(T^{t_1} P/F_1) + H(T^{t_2} / T^{t_1} P \cup F_1).$$

Therefore we have for $t_1 = 0$

$$H\Big(\bigcup_{i=1}^{2} T^{t_1} P/F_1\Big) = H(P/F_1) + H(T^{t_2} P/P \cup F_1).$$

Assume that this equality is true for n. We shall prove that it holds for $n+1$.

$$H\Big(\bigcup_{i=1}^{n+1} T^{t_i} P/F_1\Big) = H\Big(\Big(\bigcup_{i=1}^{n} T^{t_i} P \cup T^{t_{n+1}} P\Big)/F_1\Big) \text{ by (ii) of Proposition 2.2}$$

$$= H\Big(\bigcup_{i=1}^{n} T^{t_i} P/F_1\Big) + H\Big(T^{t_{n+1}} P/\bigcup_{j=1}^{n} T^{t_j} P \cup F_1\Big) \text{ by the hypothesis}$$

$$= H(P/F_1) + \sum_{i=2}^{n} H\Big(T^{t_i} P/\bigcup_{j=1}^{i-1} T^{t_j} P \cup F_1\Big) + H\Big(T^{t_{n+1}} P/\bigcup_{j=1}^{i-1} T^{t_i} P \cup F_1\Big)$$

$$= H(P/F_1) + \sum_{i=2}^{n} H\Big(T^{t_i} P/\bigcup_{j=1}^{i-1} T^{t_j} P \cup F_1\Big).$$

Dividing the above equality by $n > 0$ and taking the superior limit, the result follows from Theorem 3.4, i.e.

$$h_D(T, P/F_1) = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=2}^{n} H\left(T^{t_i} P / \bigcup_{j=1}^{i-1} T^{t_j} P \cup F_1\right).$$

3.7 Proposition. Suppose that (Y, F_1, μ_1, T_1) is a factor of fuzzy dynamical system (X, F, μ, T) and T is an invertible measure-preserving σ-homomorphism. Let P and Q be two fuzzy complete systems of X and $D = (t_n)_{n\geq 1}$ be a sequence of integers. Then

(i) $h_D(T, P/F_1) \leq h_D(T, P)$

(ii) $h_D(T, P/F_1) \leq H(P/F_1)$

(iii) If $P \subset Q$ then $h_D(T, P/F_1) \leq h_D(T, Q/F_1)$

(iv) $h_D(T, P/F_1) \leq h_D(T, Q/F_1) + H(P/Q \cup F_1)$

(v) $h_D(T, P \cup Q/F_1) \leq h_D(T, P/F_1) + h_D(T, Q/F_1)$.

Proof. (i) Since

$$\frac{1}{n} H\left(\bigcup_{i=1}^{n} T^{t_i} P/F_1\right) \leq \frac{1}{n} H\left(\bigcup_{i=1}^{n} T^{t_i} P\right) \text{ by (vi) of Proposition 2.2,}$$

we obtain thus from Theorem 3.4

$$h_D(T, P/F_1) \leq h_D(T, P),$$

where the function $h_D(T, P)$ is called the fuzzy sequence entropy function of P with respect to T.

(ii) Since

$$\frac{1}{n} H\left(\bigcup_{i=1}^{n} T^{t_i} P/F_1\right) \leq \frac{1}{n} \sum_{i=1}^{n} H(T^{t_i} P \cup F_1) \text{ by (iv) of Proposition 2.2}$$

$$= \frac{1}{n} \sum_{i=1}^{n} H(P/F_1) \text{ by (v) of Proposition 2.2}$$

$$= H(P/F_1),$$

we obtain also from Theorem 3.4

$$h_D(T, P/F_1) \leq H(P/F_1).$$

(hi) If $P \subset Q$ then $\bigcup_{i=1}^{n} T^{t_i} P \subset \bigcup_{i=1}^{n} T^{t_i} Q$.

Therefore we have from (iii) of Proposition 2.2

$$H\Big(\bigcup_{i=1}^{n} T^{t_i} P/F_1\Big) \leq H\Big(\bigcup_{i=1}^{n} T^{t_i} Q/F_1\Big).$$

Dividing the above inequality by $n>0$ and taking the superior limit when $n \to \infty$ the result follows from Theorem 3.4, i.e.

$$h_D(T, P/F_1) \leq h_D(T, Q/F_1).$$

(iv) $H\Big(\bigcup_{i=1}^{n} T^{t_i} P/F_1\Big) \leq H\Big(\Big(\bigcup_{i=1}^{n} T^{t_i} P \cup \bigcup_{i=1}^{n} T^{t_i} Q\Big)/F_1\Big)$ by (iv) of Proposition 2.2

$$\leq H\Big(\bigcup_{i=1}^{n} T^{t_i} Q/F_1\Big) + H\Big(\bigcup_{i=1}^{n} T^{t_i} P/\bigcup_{i=1}^{n} T^{t_i} Q \cup F_1\Big) \text{ by Lemma 3.6}$$

$$\leq H\Big(\bigcup_{i=1}^{n} T^{t_i} Q/F_1\Big) + \sum_{i=1}^{n} H(T^{t_i} P/T^{t_i} Q \cup F_1) \text{ by (v) of Proposition 2.2}$$

$$\leq H\Big(\bigcup_{i=1}^{n} T^{t_i} Q/F_1\Big) + n\,H(P/Q \cup F_1).$$

Dividing the above inequality by $n>0$ and taking the superior limit when $n \to \infty$, we have thus from Theorem 3.4

$$h_D(T, P/F_1) \leq h_D(T, Q/F_1) + H(P/Q \cup F_1).$$

(v) $H\Big(\Big(\bigcup_{i=1}^{n} T^{t_i} P \cup \bigcup_{i=1}^{n} T^{t_i} Q\Big)/F_1\Big)$ by (iv) of Proposition 2.2

$$\leq H\Big(\bigcup_{i=1}^{n} T^{t_i} P/F_1\Big) + H\Big(\bigcup_{i=1}^{n} T^{t_i} Q/F_1\Big).$$

Dividing the above inequality by $n>0$ and passing to the superior limit when $n \to \infty$, we obtain thus from Theorem 3.4

$$h_D(T, P \cup Q/F_1) \leq h_D(T, P/F_1) + h_D(T, Q/F_1).$$

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