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SOME PROPERTIES OF A SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Summary : The Riemannian manifold admitting a semi-symmetric metric connection have been studied by various authors ([1], [2], [3], [4], [5]). In the present paper we consider a Riemannian manifold M^n admitting a semi-symmetric metric connection ∇. We have deduced necessary and sufficient conditions for the symmetry of the Ricci tensor of a semi-symmetric metric connection and finally it is shown that the symmetry of the Ricci tensor together with the recurrent torsion tensor implies that the vector associated with the torsion tensor is a torse-forming vector field [6].

BİR RIEMANN MANİFOLDU ÜZERİNDE YARI SİMETRİK BİR METRİK BAĞLANTININ BAZI ÖZELİKLERİ

Özet : Bu çalışmada yarı simetrik bir ∇ metrik bağlantısını haiz bir M^n Riemann manifoldu gözönüne alınmakta, yarı simetrik bir metrik bağlantının Ricci tensörünün simetrisi için gerek ve yeter koşullar elde edilmekte ve son olarak, Ricci tensörünün simetrisinin "recurrent" torsiyon tensörü ile birlikte, torsiyon tensörüne ilişkin vektörün bir "torse-forming" vektör alanı olmasını gerektirdiği gösterilmektedir.

INTRODUCTION

Let M^n be an n-dimensional Riemannian manifold with Levi-Civita connection $\overline{\nabla}$. A linear connection ∇ on M^n is said to be a semi-symmetric metric connection if the torsion tensor T of the connection ∇ and the metric tensor g of the manifold satisfies the following conditions :

(1) $T(X, Y) = \omega(Y)X - \omega(X)Y$ for any two vector fields X, Y where ω is a 1-form associated with the torsion tensor of the connection ∇

and

(2) $(\nabla_Z g)(X, Y) = 0$

and further,

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(3) if $(\nabla_Z T)(X, Y) = B(Z)\, T(X, Y)$, then the torsion tensor T is said to be recurrent with B as a 1-form of recurrence.

Then we have [5] for any vector fields X, Y, Z

(4) $\nabla_X Y = \overline{\nabla}_X Y + \omega(Y) X - g(X, Y) V$,

where

(5) $g(X, V) = \omega(X)$, the 1-form ω and the vector field V are usually called 1-form and vector field associated with torsion tensor T

and

(6) $(\nabla_X \omega)(Y) = (\overline{\nabla}_X \omega)(Y) - \omega(X)\omega(Y) + \omega(V) g(X, Y)$.

Also, we have [5]

(7) $R(X, Y) Z = K(X, Y) Z - h(Y, Z) X + h(X, Z) Y - g(Y, Z) AX + g(X, Z) AY$

where

(8) $h(Y, Z) = g(AY, Z) = (\overline{\nabla}_Y \omega)(Z) - \omega(Y)\omega(Z) + 1/2\, \omega(V) g(Y, Z)$ and R and K are the respective curvature tensors for the connection ∇ and $\overline{\nabla}$, A being a (1-1) tensor field.

Further, a vector field V in a Riemannian manifold is said to be torse-forming [6] if

(9) $\overline{\nabla}_X V = \alpha X + \beta\, b(X) V$ for all X where α, β are scalars, b is a 1-form and $\overline{\nabla}$ denotes differentiation with respect to the metric of the manifold.

1. SYMMETRY CONDITION OF THE RICCI TENSOR OF ∇

In this section necessary and sufficient conditions for the symmetry of the Ricci tensor of a semi-symmetric metric connection are obtained by proving the following theorems :

Theorem 1. The Ricci-tensor $S(Y, Z)$ of a semi-symmetric metric connection ∇ with ω as its associated 1-form will be symmetric if and only if ω is closed.

Proof. Let $S(Y, Z) =$ Trace of the map : $X \rightarrow R(X, Y) Z$ where R is the curvature tensor for the connection ∇ given by (7). Then, from (7) we get

$$S(Y, Z) = \overline{S}(Y, Z) - (n-2)\, h(Y, Z) - pg(Y, Z) \tag{1.1}$$

where $\overline{S}$ denotes the Ricci tensor of the connection $\overline{\nabla}$ and p is the trace of A given by (8).

Now, from (1.1) it follows that $S(Y, Z)$ is symmetric if and only if $h(Y, Z)$ is symmetric. So, from (8) we find that the Ricci tensor $S(Y, Z)$ for the connection ∇ is symmetric if and only if $d\omega(X, Y)=0$, where d denotes exterior differentiation. That is, ω is closed. This completes the proof.

Next, we shall find another necessary and sufficient condition for which the Ricci tensor for the semi-symmetric metric connection will be symmetric.

Theorem 2. A necessary and sufficient condition that the Ricci tensor of the semi-symmetric metric connection ∇ to be symmetric is that the (0,4) curvature tensor $'R$ of the connection ∇ satisfies one of the following two conditions:

$$'R(X, Y, Z, U) = {'R}(Z, U, X, Y) \tag{1.2}$$

$$'R(X, Y, Z, U) + {'R}(Y, Z, X, U) + {'R}(Z, X, Y, U) = 0 \tag{1.3}$$

where

$$'R(X, Y, Z, U) = g(R(X, Y) Z, U). \tag{1.4}$$

Proof. From (7) we have

$$\begin{aligned} 'R(X, Y, Z, U) = {'K}(X, Y, Z, U) - h(Y, Z) g(X, U) + h(X, Z) g(Y, U) \\ - g(Y, Z) h(X, U) + g(X, Z) h(Y, U) \end{aligned} \tag{1.5}$$

where

$$h(X, Y) = g(AX, Y) = (\nabla_X \omega)(Y) - \omega(X)\omega(Y) + 1/2\, \omega(V) g(X, Y) \tag{1.6}$$

and

$$'K(X, Y, Z, U) = g(K(X, Y) Z, U).$$

From the relation (1.5), it follows that

$$\begin{aligned} 'R(X, Y, Z, U) - {'R}(Z, U, X, Y) = \\ = g(X, U)[h(Y, Z) - h(Z, Y)] + \\ + g(Z, X)[h(U, Y) - h(Y, U)] + \\ + g(Y, Z)[h(X, U) - h(U, X)] + \\ + g(Y, U)[h(Z, X) - h(X, Z)] \end{aligned} \tag{1.7}$$

for $'K(X, Y, Z, U) = {'K}(Z, U, X, Y)$ and $g(X, Y) = g(Y, X)$, $'K$ being the (0, 4) curvature tensor of the manifold.

Now $'R(X, Y, Z, U) = {'R}(Z, U, X, Y)$ if and only if

$$\begin{aligned} g(X, U)[h(Y, Z) - h(Z, Y)] + g(Z, X)[h(U, Y) - h(Y, U)] + \\ + g(Y, Z)[h(X, U) - h(U, X)] + g(Y, U)[h(Z, X) - h(X, Z)] = 0. \end{aligned} \tag{1.8}$$

Transvecting (1.8), we get

i.e.,
$$(n-2)\,[h(Y,Z)-h(Z,Y)]=0,$$
$$h(Y,Z)-h(Z,Y)=0, \tag{1.9}$$

because $n \neq 2$. Thus $'R(X,Y,Z,U) = {}'R(Z,U,X,Y)$ if and only if
$$h(Y,Z)-h(Z,Y)=0,$$

i.e., if and only if $d\omega(X,Y)=0$, i.e., if and only if Ricci tensor $S(Y,Z)$ for the connection ∇ is symmetric (by Th. 1).

Again, we have
$$\begin{aligned} &'R(X,Y,Z,U) + {}'R(Y,Z,X,U) + {}'R(Z,X,Y,U) = \\ &= g(X,Y)\,[h(Z,Y)-h(Y,Z)] + g(Z,U)\,[h(Y,X)-h(X,Y)] + \\ &+ g(Y,U)\,[h(X,Z)-h(Z,X)]. \end{aligned} \tag{1.10}$$

So,
$$'R(X,Y,Z,U) + {}'R(Y,Z,X,U) + {}'R(Z,X,Y,U) = 0$$

if and only if $h(X,Y)-h(Y,X)=0$, i.e., if and only if the Ricci tensor for the connection ∇ is symmetric. This completes the proof.

Using the above results, we now prove the following :

Theorem 3. If a Riemannian manifold M^n admits a semi-symmetric metric connection ∇ whose curvature tensor R and torsion tensor T satisfy the conditions
$$(\nabla_U R)(X,Y)Z = C(U)\,R(X,Y)Z \quad \text{and} \quad (\nabla_X T)(Y,Z)=0 \tag{1.11}$$
where C is a 1-form, then either $C(V)=2\,\omega(V)$ or $R(X,Y)Z=0$, ω being the associated 1-form of the torsion tensor for the connection ∇ such that $\omega(X)=g(X,V)$ for the vector field X.

Proof. It is known that the torsion tensor T for the semi-symmetric metric connection ∇ is given by
$$T(Y,Z)=\omega(Z)\,Y-\omega(Y)\,Z \tag{1.12}$$
where ω is a 1-form and $\omega(Y)=g(Y,V)$ for every vector field Y. Now, from the relations
$$(\nabla_X T)(Y,Z)=0 \quad \text{and} \quad (C_1^1 T)(Y)=(n-1)\,\omega(Y), \tag{1.13}$$
where C_1^1 denotes contraction we find that
$$(\nabla_X C_1^1 T)(Y)=0 \tag{1.14}$$
and
$$(\nabla_X C_1^1 T)(Y)=(n-1)\,(\nabla_X\omega)(Y). \tag{1.15}$$
Therefore, from (1.14) and (1.15) we conclude that

$$(\nabla_X \omega)(Y) = 0. \tag{1.16}$$

Hence by Ricci identity we get

$$-\omega(R(X, Y) Z) = 0, \text{ i.e., } g(R(X, Y) Z, V) = 0$$

or,

$$'R(X, Y, Z, V) = 0. \tag{1.17}$$

Again (1.16) implies that $(\overline{\nabla}_X \omega)(Y) - (\overline{\nabla}_Y \omega)(X) = 0$, i.e., $d\omega(X, Y) = 0$. Hence by Th. 1 it follows that the Ricci tensor $S(X, Y)$ is symmetric.

Therefore, by Th. 2, we find that

$$'R(X, Y, Z, V) = {}'R(Z, V, X, Y), \text{ i.e., } g(R(Z, V) X, Y) = 0 \tag{1.18}$$

for every three vector fields X, Y, Z.

Hence

$$R(Z, V) X = 0 \text{ for any two vector fields } X \text{ and } Z. \tag{1.19}$$

Again, applying second Bianchi identity for the curvature tensor R of the connection ∇, we get

$$R(T(U, X), Y) Z + R(T(X, Y), U) Z + R(T(Y, U), X) Z +$$
$$+ (\nabla_U R)(X, Y) Z + (\nabla_X R)(Y, U) Z + (A_Y R)(U, X) Z = 0.$$

From (1.11) and (1.12), we find that

$$\begin{aligned}(C(U) - 2\omega(U)) R(X, Y) Z + (C(Y) - 2\omega(Y)) R(U, X) Z +\\ + (C(X) - 2\omega(X)) R(Y, U) Z = 0\end{aligned} \tag{1.20}$$

for the vector fields X, Y, U and Z.

Putting $X = V$ in (1.20) and using (1.19) we find

$$(C(V) - 2\omega(V)) R(Y, U) Z = 0$$

for every Y, U and Z. Thus, either $C(V) = 2\omega(V)$ or $R(Y, U) Z = 0$. This completes the proof.

If, in particular, the 1-form $C = 0$ then it follows from (1.20) that $R(Y, U)Z = 0$ or $\omega(V) = 0$. If $\omega(V) = 0$, then from (5) it follows that $V = 0$, since g is positive definite. But $V = 0$ would mean that $\overline{\nabla} = \nabla$ and hence ∇ would not be semi-symmetric. Hence $R(Y, U) Z=0$. But it is known [5] that if a Riemannian manifold M^n ($n>3$) admits a semi-symmetric metric connection whose curvature tensor vanishes, then the manifold is conformally flat. Hence we can state the following corollary:

Corollary. If a Riemannian manifold M^n ($n > 3$) admits a semi-symmetric metric connection ∇ whose curvature tensor and torsion tensor are covariant constant, then the manifold is conformally flat.

2. EXISTENCE OF A TORSE-FORMING VECTOR FIELD

In this section, we consider a Riemannian manifold M^n ($n>3$) that admits a semi-symmetric connection ∇ whose Ricci tensor S is symmetric and torsion tensor is recurrent. It is shown that if a Riemannian manifold admits such a connection then the manifold admits a torse-forming vector field which was defined earlier.

It is known [1] that if a Riemannian manifold admits a semi-symmetric metric connection ∇ with recurrent torsion tensor T given by

$$T(X, Y) = \omega(Y) X - \omega(X) Y \tag{2.1}$$

where $\omega(X) = g(X, V)$ and

$$(\nabla_Z T)(X, Y) = B(Z) T(X, Y) \tag{2.2}$$

where B is a 1-form, then the curvature tensor R for the connection ∇ is given by

$$\begin{aligned} R(X, Y) Z = K(X, Y) Z &+ B(X) [\omega(Z) Y - g(Y, Z) V] + \\ &+ B(Y) [g(X, Z) V - \omega(Z) X] + \\ &+ \omega(V) [g(Y, Z) X - g(X, Z) Y]. \end{aligned} \tag{2.3}$$

From (2.1) we have

$$(C_1^1 T)(Y) = (n-1) \omega(Y). \tag{2.4}$$

From (2.4) we get

$$(\nabla_X C_1^1 T)(Y) = (n-1)(\nabla_X \omega)(Y). \tag{2.5}$$

Again from (2.2) we obtain

$$\begin{aligned} (\nabla_X C_1^1 T)(Y) &= B(X)(C_1^1 T)(Y) = B(X)(n-1)\omega(Y) \\ &= (n-1) B(X) \omega(Y). \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we get

$$(\nabla_X \omega)(Y) = B(X) \omega(Y). \tag{2.7}$$

Now, from (2.3) and from the fact that

$$S(Y, Z) = \text{trace of the map}: X \to R(X, Y) Z$$

we find that

$$\begin{aligned} S(Y, Z) = \bar{S}(Y, Z) &- (n-2) B(Y) \omega(Z) - \\ &- (B(V) - (n-1)\omega(V)) g(Y, Z). \end{aligned} \tag{2.8}$$

Since the Ricci tensor S for the connection ∇ is symmetric, we deduce that

$$B(Y) \omega(Z) = B(Z) \omega(Y). \tag{2.9}$$

Now, putting $Z = V$ in (2.9) we get

$$B(Y) = a\,\omega(Y) \tag{2.10}$$

where a is determined by

$$B(V) = a\,\omega(V).$$

So, (2.7) takes the form

$$(\nabla_X \omega)(Y) = a\;\omega(X)\,\omega(Y). \tag{2.11}$$

Therefore, from (6) and (2.11) it is seen that

$$(\overline{\nabla}_X \omega)(Y) = \alpha\, g(X, Y) + \beta\, \omega(X)\, \omega(Y) \tag{2.12}$$

where

$$\alpha = -\,\omega(V) \text{ and } \beta = a + 1. \tag{2.13}$$

Thus, we get the following :

Theorem 4. If a Riemannian manifold admits a semi-symmetric metric connection ∇ with symmetric Ricci tensor and recurrent torsion tensor, then the manifold always admits a torse-forming vector field.

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