BOUNDED AND ALMOST PERIODICAL SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF SECOND ORDER

M.B. RAGIMOV

370149 Baku-149, Taganrogskaya Street, 31, flat 9, Azerbaijan Republic

Summary : In this paper we consider the linear differential equation of the second order of the following type:

$$A\ddot{x} + B\dot{x} + Cx = \psi(t) \qquad (0.1)$$

where $A : D(A) \subset X \to X$, $B : D(B) \subset X \to X$, $C : D(C) \subset X \to X$ are linear operators, acting in complex Banach space X and having the definition domains $D(A)$, $D(B)$, $D(C)$ resp. such that the subspace $X_0 = D(A) \cap \cap D(B) \cap D(C)$ is dense in X. We make various assumptions about the function $\psi : (a, b) \to X$. Nevertheless one can obtain the most important results in that case, where ψ belongs to the Banach space $\mathbf{C}(\mathbf{R}; X)$ of the X-valued continuous functions on $\mathbf{R}$ and especially to the subspace of continuous almost periodical functions [1].

The case $A, B, C \in L(X; Y)$ and $\psi \in B(\mathbf{R}; Y)$ is considered separately. The equation (0.1) for $\psi \in \mathbf{C}(\mathbf{R}; X)$ is conveniently to consider as the operator equation of the following type

$$Lx = \psi,$$

where $L : D(L) \subset \mathbf{C}(\mathbf{R}; X) \to \mathbf{C}(\mathbf{R}; X)$ is linear operator

$$Lx \equiv A\ddot{x} + B\dot{x} + Cx,$$

with appropriate definition domain $D(L)$.

İKİNCİ MERTEBEDEN DİFERANSİYEL DENKLEMLERİN SINIRLI VE HEMEN HEMEN PERİYODİK ÇÖZÜMLERİ

Özet : Bu çalışmada

$$A\ddot{x} + B\dot{x} + Cx = \psi(t) \qquad (0.1)$$

tipindeki 2. mertebeden lineer diferansiyel denklem gözönüne alınmaktadır ki, burada $A : D(A) \subset X \to X$, $B : D(B) \subset X \to X$, $C : D(C) \subset X \to X$, kompleks X Banach uzayında etki eden ve tanım bölgeleri sırasıyla $D(A)$, $D(B)$, $D(C)$ olan lineer operatörlerdir ve $X_0 = D(A) \cap D(B) \cap D(C)$ alt uzayı, X te yoğundur. $\psi : (a, b) \to X$ fonksiyonu ile ilgili çeşitli varsayımlar yapılmakta ve en önemli sonuçlar, ψ nin, $\mathbf{R}$ üzerindeki X değerli sürekli fonksiyonların $\mathbf{C}(\mathbf{R}; X)$ Banach uzayına ve özellikle, sürekli ve hemen hemen periyodik fonksiyonlardan oluşan alt uzaya [1] ait olması durumunda

elde edilmektedir. $A, B, C \in L(X; Y)$ ve $\psi \in B(\mathbf{R}; Y)$ halleri ayrı ayrı gözönüne alınmaktadır. $\psi \in C(\mathbf{R}; X)$ için (0.1) denklemi, $LX = \psi$ tipindeki operatör denklemi olarak düşünülmektedir ki, burada $L : D(L) \subset C(\mathbf{R}; X) \to C(\mathbf{R}; X)$, uygun bir $D(L)$ tanım bölgesi ile,

$$Lx \equiv A\ddot{x} + B\dot{x} = Cx$$

lineer operatörüdür.

§1. On the definition domain of differential operator

The problem to determine the definition domain $D(L)$ is every complex in the case when at least one of A, B, C is non-bounded operator.

To determine $D(L)$ one ought to make some additional assumptions on the operators A, B, C considering the operator-valued function H, defined on $\mathbf{R}$, with values in the set of linear closed operators. This function is determined by the formula $H(\lambda) = -\lambda^2 A + iB\lambda + C$ under the assumption that each of operators $H(\lambda) : D(H(\lambda)) \subset X \to X$ is closed extension with $X_0 = D(A) \cap D(B) \cap D(C)$ and $D(H(\lambda)) \supset D(A)$, $D(H(\lambda)) \supset D(B)$. Let us make the main assumptions respected to the bundle of operators $H(\lambda)$, $\lambda \in \mathbf{C}$: There exists such complex number Z_0, that the operator-valued function $H_0(\lambda) = H(\lambda) + Z_0 I$ satisfies the conditions:

1) Each of operators $H_0(\lambda)$, $\lambda \in \mathbf{R}$ has a continuous inverse and the inverse $H_0^{-1}(\lambda)$ admits the estimation of the following type :

$$\| H_0^{-1}(\lambda) \| \leq \frac{\text{const}}{(1 + |\lambda|)^{1+\alpha}}, \quad \forall \in \mathbf{R} \tag{1.1}$$

where $\alpha > 0$ is some number and const is absolute constant, independent on $\lambda \in \mathbf{R}$;

2) Operators $A H_0^{-1}(\lambda)$, $B H_0^{-1}(\lambda)$ are bounded and

$$\| A H_0^{-1}(\lambda) \| \leq \frac{\text{const}}{(1 + |\lambda|)}, \quad \| B H_0^{-1}(\lambda) \| \leq \text{const}. \tag{1.2}$$

for each $\lambda \in \mathbf{R}$ (const -the absolute constant, independent on $\lambda \in \mathbf{R}$. The trying of the conditions (1.1)-(1.2) is simple than the tryingof the correctness condition of the Cauchy problem for respective homogeneous equation (0.1) ($\Psi = 0$). The correctness conditions of the Cauchy problem are offered in [2], [3], [4].

Note 1.1. In the case $A, B, C \in L(X; Y)$ the respective function

$$H : \mathbf{R} \to L(X; Y)$$

is defined as : $H(\lambda) = -\lambda^2 A + i\lambda B + C$ and about this function one ought to make the following assumption, different from above ones:

Suppose that the space X is continuously imbedded into Y and there exists $Z_0 \in \mathbf{C}$ such that the function $H_0(\lambda) = H(\lambda) + Z_0 I_*$, where the imbedding operator $I_* : X \to Y$ has continuous inverse for each $\lambda \in \mathbf{R}$ and analogs of the conditions (1.1)-(1.2) hold

$$\| H_0^{-1}(\lambda) \| \leq \frac{\text{const}}{(1 + |\lambda|)^{1+\alpha}}, \quad \alpha > 0 \tag{1.1$'$}$$

$$\| A H_0^{-1}(\lambda) \| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \| B H_0^{-1}(\lambda) \| \leq \text{const}, \tag{1.2$'$}$$

where $\lambda \in \mathbf{R}$ and const is absolute constant.

Lemma 1.1. The function $H_0(\lambda)$ satisfying the consditions (1.1)-(1.2) (resp. $H_0 : \mathbf{R} \to L(X ; Y)$ satisfying the conditions (1.1)$'$-(1.2)$'$) has the property: there exists the continuous operator-valued summable function $G : \mathbf{R} \to L(X)$ (resp. $G : \mathbf{R} \to L(Y ; X)$) such that

$$H_0^{-1}(\lambda) = \int_{-\infty}^{\infty} G(t)\, e^{-i\lambda t}\, dt, \quad \lambda \in \mathbf{R}$$

i.e., the function $H_0^{-1}(\lambda)$ is the Fourier transformation of the summable function $G(t) \left(\int_{-\infty}^{\infty} \| G(t) \|\, dt < +\infty \right)$.

Proof. Set

$$G(t) = \int_{-\infty}^{\infty} H_0^{-1}(\lambda)\, e^{i\lambda t}\, d\lambda, \quad t \in \mathbf{R}. \tag{1.3}$$

From the condition of summability of the function $H_0^{-1}(\lambda)$ (which follows from the condition (1.1) or (1.1)$'$) it follows that this function is continuous and bounded (moreover, $\| G(t) \| \to 0$ as $|t| \to \infty$). From the definition of the function $H_0^{-1}(\lambda)$ and condition (1.2) or (1.2)$'$ it follows immediately that

$$\frac{d H_0^{-1}(\lambda)}{d\lambda} = H_0^{-1}(\lambda)\,(-2 A \lambda + i B)\, H_0^{-1}(\lambda)$$

and

$$\left\| \frac{d H_0^{-1}(\lambda)}{d\lambda} \right\| \leq \text{const} \, \| H_0^{-1}(\lambda) \| \left(\frac{|\lambda|}{1 + |\lambda|} + 1 \right) \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \mathbf{R},$$

i.e., the function $H_0^{-1}(\lambda)$ is continuously differentiable and summable on $\mathbf{R}$.

One can analogously prove that it has the second derivative $\dfrac{d^2 H_0^{-1}}{d\lambda^2}$ and

$$\left\| \frac{d^2 H_0^{-1}(\lambda)}{d\lambda^2} \right\| \leq \frac{\text{const}}{(1+|\lambda|)^\alpha}.$$

Taking it into account from (1.3) one can obtain the equality

$$-t^2 G(t) = \int_{-\infty}^{\infty} \frac{d^2 H_0^{-1}(\lambda)}{d\lambda^2} e^{i\lambda t}\, d\lambda,$$

from the condition $\operatorname{Sup}_{t\in\mathbf{R}} \| t^2 G(t) \| < +\infty$ it follows that $\| G(t) \| \leq \frac{c}{t^2}$. Since this function is continuous, it is bounded in the neighbourhood of 0 and finally $\int_{-\infty}^{\infty} \| G(t) \|\, dt < +\infty$.

Note 1.2. From the definition, the functions of the type

$$H_0^{-1}(\lambda)\, x_0\, e^{i\lambda_0 t} = \int_{-\infty}^{\infty} G(t)\, x_0\, e^{i\lambda_0 t}\, dt$$

belong to $D(L)$ and therefore from this equality it follows that the function $y_0 e^{i\lambda t}$ belongs to $D(L)$ if $y_0 \in D(H_0(\lambda_0))$. Since $D(H_0(\lambda)) = D(H(\lambda_0))$, the function $y_0 e^{i\lambda_0 t}$ belongs to $D(L)$ if $y_0 \in D(H(\lambda_0))$.

Definition 1.1. Continuous bounded function $\varphi : \mathbf{R} \to X$ is called the generalized bounded solution of the equation (I) if it satisfies the equation

$$\varphi(t) = \int_{-\infty}^{\infty} G(t-s)\,(\Psi(s) + Z_0\, \varphi(s))\, ds.$$

This approach to the definition of generalized solution of the equation (I) is very convenient from operator viewpoint. The main reason of it is that the conditions (1.1)-(1.2) on bundle make possible to define the natural definition domain $D(L)$ of the operator $L : D(L) \subset C(\mathbf{R}\,; X) \to C(\mathbf{R}\,; X)$ in the Banach space $C(\mathbf{R}\,; X)$ of the functions bounded on $\mathbf{R}$ with values on X, defined by the differential expression :

$$Lx \equiv A\ddot{x} + B\dot{x} + Cx.$$

In order to define $D(L)$ we proceed as follows : firstly we define the definition domain of the operator $L + Z_0 I$ and then write: $D(L) = D(L + Z_0 I)$.

§2. On the almost non periodicity set of bounded functions

For the investigation of almost periodical (generalized) solutions of the equations we will proceed making use of the notions of Börling spectrum and the almost non-periodicity set of bounded functions.

Definition 2.1. The Börling spectrum $S(\varphi)$ of the continuous bounded function $\varphi : \mathbf{R} \to X$ ($\varphi \in C(\mathbf{R}; X)$) is referred to the common zeros set of the Fourier transformation of the functions from the set

$$\{f \in L_1(\mathbf{R}) : f * \varphi = 0\},$$

where $L_1(\mathbf{R})$ is the Banach algebra of the complex-valued functions summable on $\mathbf{R}$ with convolution as the product of functions

$$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(t-\sigma) f_2(\sigma)\, d\sigma, \quad f_1, f_2 \in L_1(\mathbf{R}).$$

The convolution of the functions $f \in L_1(\mathbf{R})$, $\varphi \in C(\mathbf{R}; X)$ is referred to the function from $C(\mathbf{R}; X)$, defined as follows :

$$(f * \varphi)(t) = \int_{-\infty}^{\infty} f(t-\sigma)\, \varphi(\sigma)\, d\sigma.$$

Note the property $\|f*\varphi\|_C \leq \|f\|_1 \cdot \|\varphi\|_C$, where $\|\cdot\|_C$ is the norm in $C(\mathbf{R}; X)$ and $\|f\|_1 = \int_{-\infty}^{\infty} \|f(t)\|\, dt$ is the norm in $L_1(\mathbf{R})$.

From the definition of $S(\varphi)$ immediately follows that $\lambda_0 \overline{\in} S(\varphi)$ if there exists the function $f \in L_1(\mathbf{R})$ with properties $\widehat{f}(\lambda_0) \neq 0$ and $f * \varphi = 0$.

From the definition (2.1) also follows that the Börling spectrum $S(\varphi)$ of the function $\varphi \in C(\mathbf{R}; X)$ is equal to the support of the Fourier transformation of the function φ considering as generalized function of slow growth. This allows to formulate the properties of the Börling spectrum. analogous to the property of support of the functions.

Lemma 2.1. Börling spectrum of the functions from $C(\mathbf{R}; X)$ has the following properties :

1) $S(\varphi)$ is closed and $S(\varphi) = \phi$, if $\varphi = 0$;
2) $S(f_0 * \varphi) \subset \mathrm{Supp}\,\widehat{f}_0 \cap S(\varphi)$ $\forall f_0 \in L_1(\mathbf{R})$, $\forall \varphi \in C$; (supp f_0-support of the function f_0);
3) $f_0 * \varphi = 0$, if $\mathrm{supp}\,\widehat{f}_0 \cap S(\varphi) = \phi$;
4) $(\alpha\varphi + f * \varphi) = \varphi$, if the function $\widehat{f}(\lambda) + \alpha$ is equal to I in the neighborhood of the set $S(\varphi)$ ($f \in L_1(\mathbf{R})$, $\alpha \in \mathbf{C}$);
5) $S(\varphi + \Psi) \subset S(\varphi) \cup S(\Psi)$ $\forall \varphi, \Psi \in \mathbf{C}$;
6) $S(\varphi) = \overline{\sigma(\varphi)}$, if $\varphi \in B(\mathbf{R}; X)$, i.e. φ is almost periodical function with the Fourier series of the kind

$$\varphi(t) \sim \sum_{j=1}^{\infty} \varphi_j e^{i\lambda_j t}, \ \varphi_j \in X, \ \sigma(\varphi) = \{\lambda \in \mathbf{R} : \varphi_j \neq 0\}.$$

Proof. 1) It is clear that $f * \varphi = 0 \ \forall f \in L_1(\mathbf{R})$ provided that $\varphi = 0$ and therefore the common set of zeroes of the Fourier transforms from $L_1(\mathbf{R})$ which is equal to $S(\varphi)$ is empty. So $S(\varphi) = \phi$. Conversely, let $S(\varphi) = \phi$, i.e. the common set of zeroes of the Fourier transforms of the function from $M(\varphi) = \{f \in L_1(\mathbf{R}) : f * \varphi = 0\}$ is empty. Since the set $M(\varphi)$ is invariant with respect to functions shifts, by the Viner's taubers theorem, we obtain $M(\varphi) = L_1(\mathbf{R})$, i.e. $f * \varphi = 0$, $f \in L_1(\mathbf{R})$. From this it follows that $\varphi = 0$. The closedness of the set $S(\varphi)$ is obvious.

2) Let $f_0 \in L_1(\mathbf{R})$ and $\varphi \in C(\mathbf{R}; X)$. If $\lambda_0 \overline{\in} \operatorname{Supp} \widehat{f_0}$, then $\widehat{f}(\lambda_0) = 0$ on certain interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for some $\varepsilon > 0$. Select the function $f \in L_1(\mathbf{R})$ such that $\operatorname{Supp} \widehat{f} \subset (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and $\widehat{f}(\lambda_0) \neq 0$. Then $f * (f_0 * \varphi) = (f * f_0) * \varphi = 0$. For $\widehat{f}(\lambda_0) = 0$, $\lambda_0 \overline{\in} S(f * \varphi)$, $S(f * \varphi) \subset \operatorname{Supp} \widehat{f}$. If $\lambda_0 \overline{\in} S(\varphi)$ then there exists a function $g \in L_1(\mathbf{R})$ with properties: $\widehat{g}(\lambda_0) \neq 0$, $g * \varphi = 0$. Then $g * (f * \varphi) = f * (g * \varphi) = 0$, $\lambda_0 \overline{\in} S(f * \varphi)$, i.e. $S(f * \varphi) \subset S(\varphi)$, finally $S(f * \varphi) \subset \operatorname{Supp} \widehat{f} \cap \cap S(\varphi)$.

3) From 2) it follows that $S(f * \varphi) \subset \operatorname{Supp} \widehat{f} \cap S(\varphi) = \phi$, $S(f * \varphi) = \phi$ and therefore from 1) it follows that $f * \varphi = 0$.

4) Suppose that $\alpha \in C$ and the function $f \in L_1(\mathbf{R})$ is selected such that $\widehat{f}(\lambda) + \alpha = 1$ in the neighborhood of the set $S(\varphi)$. Consider the function $\Psi = \alpha \varphi + f * \varphi - \varphi$. Then for each function $g \in L_1(\mathbf{R})$ we obtain the equality $g * \Psi = \alpha g * \varphi + g * f * \varphi - g * \varphi = (\alpha g + g * f - g) * \varphi = f_0 * \varphi$. Since $\widehat{f_0}(\lambda) = \alpha \widehat{g}(\lambda) + \widehat{g}(\lambda) \widehat{f}(\lambda) - \widehat{g}(\lambda) = \widehat{g}(\lambda) (\alpha + \widehat{f}(\lambda) - 1) = 0$ in the neighborhood of $S(\varphi)$, then from (2) it follows that $f_0 * \varphi = 0$, i.e. $g * \Psi = 0 \ \forall g \in L_1(\mathbf{R})$ and therefore the function $\Psi = \alpha \varphi + f * \varphi - \varphi$ is equal to zero, i.e. $\alpha \varphi + f * \varphi = \varphi$.

5) If $\lambda_0 \in S(\varphi) \cup S(\Psi)$, then there exist functions $f_1, f_2 \in L_1(\mathbf{R})$ such that $\widehat{f_1}(\lambda) \neq 0$, $\widehat{f_2}(\lambda) \neq 0$, $f_1 * \varphi = f_2 * \varphi$. Then the function $f = f_1 * f_2$ will have the properties $\widehat{f}(\lambda_0) = \widehat{f_1}(\lambda_0) \widehat{f_2}(\lambda_0) \neq 0$, and $f * (\varphi + \Psi) = f_1 * f_2 (\varphi + \Psi) = f_2 * (f_1 * \varphi) + f_1 * (f_2 * \varphi) = 0$, i.e. $\lambda_0 \overline{\in} S(\varphi + \Psi)$.

6) Suppose that $\varphi \in B(\mathbf{R}; X)$ with the Fourier series

$$\varphi(t) \sim \sum_{j=1}^{\infty} \varphi_j e^{i\lambda_j t}.$$

Then for each function $f \in L_1(\mathbf{R})$ one can immediately find from the definition that $f * \varphi$ is again the almost periodical function and it has the Fourier series

$$f * \varphi \sim \sum_{j=1}^{\infty} \hat{f}(\lambda_j)\, \varphi_j\, e^{i\lambda_j t}.$$

From this type of Fourier series for the function $f * \varphi$ it follows that if $\varphi_j \neq 0$ and $\hat{f}(\lambda_j) \neq 0$ (i.e. $\lambda_j \in \sigma(\varphi)$) then $f * \varphi \neq 0$ and therefore $\lambda_j \in S(\varphi)$, i.e. $\sigma(\varphi) \subset S(\varphi)$, and therefore $\overline{\sigma(\varphi)} \subset S(\varphi)$ ($\sigma(\varphi)$ may not be closed set). If $\lambda_0 \overline{\in} \overline{\sigma(\varphi)}$, then select the function $f \in L_1(\mathbf{R})$ with properties $\hat{f}(\lambda_0) \neq 0$ and supp $\hat{f} \cap \overline{\sigma(\varphi)} = \phi$. Then

$$f * \varphi \sim \sum \hat{f}(\lambda_j)\, \varphi_j\, e^{i\lambda_j t} = 0$$

and therefore by the uniqueness theorem about Fourier series $f * \varphi = 0$; so $\lambda_0 \overline{\in} S(\varphi)$.

Definition 2.2. By the set of non almost periodicity of the bounded continuous function $\varphi : \mathbf{R} \to X$ we call the common set of zeroes of the Fourier transformations of the functions from the set $\{f \in L_1(\mathbf{R}) : f * \varphi$ -almost periodical function$\}$. The set of non-almost periodicity of the function φ is denoted by $S_0(\varphi)$. Note that $\lambda_0 \overline{\in} S_0(\varphi)$ iff there exists the function $f_0 \in L_1(\mathbb{R})$ with properties: $\hat{f}_0(\lambda_0) \neq 0$ and $f_0 * \varphi \in B(\mathbf{R}\,;X)$. In the following lemma we formulate some properties of the set of non almost periodicity of functions, which are analogous to the properties of the Börling spectrum of functions.

Lemma 2.2. The set of almost periodicity of the functions from $C(\mathbf{R}\,;X)$ has the following properties :

1) $S_0(\varphi) \subset S(\varphi)$, $\varphi \in C(\mathbb{R}\,;X)$;

2) $S_0(\varphi)$ is closed set and $S_0(\varphi) = \phi$ iff φ is almost periodical;

3) $S_0(f_0 * \varphi) \subset \operatorname{Supp} \hat{f}_0 \cap S_0(\varphi)$, $\forall f_0 \in L_1(\mathbb{R})$, $\forall \varphi \in C(\mathbf{R}\,;X)$;

4) $f * \varphi \in B(\mathbb{R}\,;X)$ if $\operatorname{Supp} \hat{f} \cap S_0(\varphi) = \phi$;

5) $S_0(\varphi + \Psi) \subset S_0(\varphi) \cup S_0(\Psi)$, $\varphi, \Psi \in C(\mathbf{R}\,;X)$.

The proof of the indicated properties is analogous to the proof of the respective properties of the Börling spectrum, which formulated in Lemma 2.1. For instance, prove the property 5). If $\lambda_0 \overline{\in} S_0(\varphi) \cup S_0(\Psi)$, then there exist two functions $f_1, f_2 \in L_1(\mathbb{R})$ with properties: $\hat{f}_1(\lambda_0) \neq 0$, $\hat{f}_2(\lambda_0) \neq 0$, $f_1 * \varphi = f_2 * \Psi \in B(\mathbf{R}\,;X)$. Then the function $f = f_1 * f_2$ has the properties $\hat{f}(\lambda_0) = \hat{f}_1(\lambda_0)\, \hat{f}_2(\lambda_0) \neq 0$, $f * (\varphi + \Psi) = f_1 * f_2 * \varphi + f_1 * f_2 * \Psi = f_2 * (f_1 * \varphi) + f_1 * (f_2 * \Psi) \in B(\mathbb{R}\,;X)$.

§3. Almost periodicity of bounded solutions

Now return to the investigation of the differential equations (1) and offer some sufficient conditions of almost periodicity of generalized bounded solution of this equation (under the condition (1)-(1.2) on operator-valued function $H(\lambda): D(H(\lambda)) \subset X \to X$ ($H: X \to Y$, if we consider the case of various spaces).

Definition 3.1. The singular set of the function $H(\lambda) = -A\lambda^2 + B(i\lambda) + C: D(H(\lambda)) \subset X \to X$, $\lambda \in \mathbf{R}$ ($H(\lambda): X \to Y$) is the complement in $\mathbf{R}$ of the set $\{\lambda \in \mathbf{R} : H(\lambda)$ has continuous inverse$\}$. Taking into account the openness of the set of invertible operations (in the case of various spaces X and Y) we obtain, that the singular set of the function H is closed. We denote it by $S(H)$, so $S(H) \subset \mathbf{R}$. For the case of one space X we will demand the closedness of the set $S(H)$.

Definition 3.2. The sequence of functions $(\varphi_1, \varphi_2, \ldots)$ from $C(\mathbf{R}; X)$ is called c-convergent to the function $\varphi \in C(\mathbf{R}; X)$ if it is bounded and uniformly converges to φ on each finite interval from $\mathbf{R}$, i.e. $\sup \| \varphi_n \| < \infty$, $\| \varphi_n - \varphi \| \to 0$ for each interval $[a, b]$. In this case we use the notation $\varphi_n \xrightarrow{c} \varphi$ or $c\text{-}\lim\limits_{n} \varphi_n = \varphi$.

Definition 3.3. The linear operator $L: D(L) \subset C(\mathbf{R}; X) \to C(\mathbf{R}; X)$ is called c-closed, if from the conditions $\varphi_n \to \varphi$, $\varphi_n \in D(L)$ and $L\varphi_n \xrightarrow{c} \Psi$ it follows that $\varphi \in D(L)$ and $L\varphi = \Psi$. The bounded operator $L: C(\mathbf{R}; X) \to C(\mathbf{R}; X)$ is called c-continuous, if from $\varphi_n \xrightarrow{c} \varphi$ it follows that $L\varphi_n \xrightarrow{c} L\varphi$. Analogous definitions are offered if $L: D(L) \subset C(\mathbf{R}; Y) \to C(\mathbf{R}; X)$ (in the case of various spaces).

Lemma 3.1. If the resolvent set $\rho(L)$ of the operator $L: D(L) \subset C(\mathbf{R}; X) \to C(\mathbf{R}; X)$ is non-void and $(L - \lambda_0 I)^{-1}: C \to C$ is c-continuous operator, then L is c-closed operator.

Proof. Let $\lambda_0 \in \rho(L)$ and the operator $L_0 = L - \lambda_0 I$ is c-continuous. Prove that the operator L is c-closed. Let $\varphi_n \xrightarrow{c} \varphi$ and $L_0 \varphi_n \xrightarrow{c} \Psi$. Then $L_0^{-1}(L_0 \varphi_n) \xrightarrow{c} L_0^{-1} \Psi$, i.e. $\varphi_n \xrightarrow{c} L_0^{-1} \Psi$. Therefore $\varphi = L_0^{-1} \Psi$, i.e. $\varphi \in D(L_0)$ and $L_0 \varphi = \Psi$.

Since $L = L_0 + \lambda I$, if $\varphi_n \xrightarrow{c} \varphi$ and $L\varphi_n \to \Psi$, then $L_0 \varphi_n \xrightarrow{c} \Psi - \lambda_0 \varphi$. Therefore $\varphi \in D(L_0) = D(L)$ and $L_0 \varphi = \Psi - \lambda_0 \varphi$, i.e. $L\varphi = \Psi$.

Theorem 3.1. The linear operator

$$L: D(L) \subset C(\mathbf{R}; X) \to C(\mathbf{R}; X) \quad (Lx \equiv A\ddot{x} + B\dot{x} + Cx)$$

is c-closed.

Proof. As we assumed there exists a complex number Z_0 such that $H_0(\lambda)=H(\lambda)+Z_0 I$, $\lambda \in \mathbf{R}$ is invertible and for the function $H_0^{-1}(\lambda)$ hold the conditions (1.1), (1.2). Following Lemma 1.1 and definition (generalized) of solution we obtain that the operator $L_0 = L + Z_0 I$ is continuously invertible and the inverse one $L_0^{-1} : C \to C$ has the form

$$(L_0^{-1}\varphi)(t) = (G * \varphi)(t) = \int_{-\infty}^{\infty} G(t-s)\,\varphi(s)\,ds,$$

where the Fourier transform $\widehat{G} : \mathbf{R} \to L(X)$ of the function G coinsides with the function $H_0^{-1}(\lambda)$, $\lambda \in \mathbf{R}$. If $\varphi_n \xrightarrow{c} \varphi$ then

$$L_0^{-1}\varphi_n - L_0^{-1}\varphi = \int_{-\infty}^{\infty} G(t-s)\,(\varphi_n(s) - \varphi(s))\,ds$$

and for all $t \in [a, b]$ $(a < b)$ we obtain the estimations

$$\| L_0^{-1}\varphi_n(t) - L_0^{-1}\varphi(t) \| \leq \int_{-\infty}^{\infty} \| G(t-s)\,(\varphi_n(s) - \varphi(s) \|\,ds =$$

$$= \int_{-N}^{N} f(t-s)\,\|\varphi_n(s) - \varphi(s)\|\,ds + \int_{|s| \geq N} f(t-s)\,\|\varphi_n(s) - \varphi(s)\|\,ds < \varepsilon,$$

for each $t \in [a, b]$ and sufficiently large N, and each given $\varepsilon > 0$, $f(t) = \| G(t) \|$. Therefore, L_0^{-1} is c-continuous operator. Then, according to previous lemma L is c-closed.

Lemma 3.2. The operator $L : D(L) \subset C(\mathbf{R}; X) \to C(\mathbf{R}; X)$ commutes with shift operators of functions, i.e. if $\varphi \in D(L)$ then for each $h \in \mathbf{R}$ the function $\varphi_h(t) = \varphi(t+h)$, $t \in \mathbf{R}$ belongs to $D(L)$ and $L\varphi_h = (L\varphi)_h$. Moreover, for each function $f \in L_1(\mathbf{R})$ and $\varphi \in D(L)$ the function $f * \varphi$ belongs to $D(L)$ and $L(f * \varphi) = f * L\varphi$.

Proof. Consider the operator $L_0 = L + Z_0 I$ which is invertible and its inverse has the form

$$(L_0^{-1}\varphi)(t) = \int_{-\infty}^{\infty} G(t-s)\,\varphi(s)\,ds = (G * \varphi)(t),$$

i.e. it is the operator of convolution with the summable function $G(t)$ which was introduced in Lemma 1.1. The operator L_0^{-1} commutes with shift operators, i.e. if $\tau_h\varphi = \varphi_h$, $\varphi \in C$, $h \in \mathbf{R}$ then

$$L_0^{-1}\tau_h\varphi(t)=\int_{-\infty}^{\infty}G(t-s)\,\varphi(s+h)\,ds=\int_{-\infty}^{\infty}G(t+h-s)\,\varphi(s)\,ds=(\tau_h L_0^{-1}\varphi)(t).$$

Therefore $L_0^{-1}\tau_h=\tau_h L_0^{-1}$. From this it follows that if $\varphi\in D(L)=D(L_0)$, then there exists $\Psi\in C$ such that $\varphi=L_0^{-1}\Psi$. Then $L_0^{-1}\tau_h\Psi=\tau_h\varphi\in D(L_0)=D(L)$ and besides it $L_0^{-1}(\tau_h\varphi)=\tau_h(L_0\varphi)$. Then $L(\tau_h\varphi)=\tau_h L\varphi$. Let us suppose that $f\in L_1(\mathbb{R})$. Then from the type of the operator $L_0^{-1}\{L_0\varphi\}=G\star\varphi$ we obtain that $f\star L_0^{-1}\varphi=L_0^{-1}(f\star\varphi)$, i.e. the operator L_0^{-1} commutes with the operator of convolution with any function from $L_1(\mathbb{R})$. From this equation it follows that if $\varphi\in D(L_0)=D(L)$ then $f\star\varphi\in D(L)$ and $L(f\star\varphi)=f\star L\varphi$, $\varphi\in D(L)$ (which can be proved completely analogously as the proof of commutability of the operator L with operators τ_h, $h\in\mathbb{R}$.

Lemma 3.3. The set of trigonometrical polynomials (i.e. functions of the type $f(t)=\sum_{j=1}^{N}x_j e^{i\lambda_j t}$, $x_j\in X$) is c-densed in $D(L)$, i.e. for each function $\varphi\in D(L)$ there exists a sequence of trigonometrical polynomials φ_j from $D(L)$ such that

$$\varphi_j\xrightarrow{c}\varphi.$$

Proof. Consider the arbitrary function $\varphi\in D(L)=D(L_0)$ $(L_0=L+Z_0 I)$ and the function $\Psi\in C$ such that $L_0^{-1}\Psi=\varphi$. Consider the arbitrary sequence of trigonometrical polynomials (φ_n) such that $\varphi_n\xrightarrow{c}\varphi$ in C. Then

$$L_0^{-1}\Psi_n\xrightarrow{c}L_0^{-1}\Psi=\varphi.$$

Since $L_0^{-1}(x_0e^{i\lambda_0 t})=\int_{-\infty}^{\infty}G(t-s)\,x_0e^{i\lambda_0 t}\,ds=G(\lambda_0)\,x_0e^{i\lambda_0 t}=H_0^{-1}(\lambda_0)\,x_0e^{i\lambda_0 t}$,

$\varphi_n=L_0^{-1}(\Psi_n)$ are trigonometrical polynomials c-converged to $L_0^{-1}\Psi=\varphi$.

Before the formulating of one of the main results we formulate one additional result about the conditions for almost periodicity for vector-functions (spectral almost-periodicity condition). The scalar case was investigated by [5], the vector one- in [6].

Theorem 3.2. Let $\varphi:\mathbb{R}\to X$ is uniformly continuous bounded function defined on $\mathbb{R}$ with values in Banach space X and the set $S_0(\varphi)$ is countable. Then φ is almost-periodical if one of the following conditions holds :

a) The space X does not contain the subspaces, isomorphic to the space C_0 of the sequences of complex numbers converged to zero;

b) the set of the values of the function φ is weakly compact;

c) $S(\varphi)$ has no limit points on finite intervals on $\mathbf{R}$.

Theorem 3.3. Let $\varphi \in C(\mathbf{R}; X)$ -(generalized) solution of the equation

$$Lx = A\ddot{x} + B\dot{x} + Cx = \Psi(t),$$

where $\Psi : \mathbf{R} \to X$ -almost periodical function. The the set of none almost periodicity $S_0(\varphi)$ belongs to the set $S(H)$ and $S(\varphi) \subset S(H) \cup S(\Psi)$.

Proof. Let $\lambda_0 \in \mathbf{R} \setminus S(H)$ and then the operators $H(\lambda)$, $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, where ε is a positiv number, are continuously invertible. We represent these operators in the form $H(\lambda) = H_0(\lambda) - Z_0 I$, where $H_0(\mu)$ is previously introduced function, which is invertible for each $\mu \in \mathbf{R}$. We can represent the function $H(\lambda)$ as follows:

$$H(\lambda) = H_0(\lambda)\ (I - Z_0 H_0^{-1}(\lambda)),\ \lambda \in (\lambda_0 - \varepsilon,\ \lambda_0 + \varepsilon).$$

Since $H(\lambda)$ and $H_0(\lambda)$ are invertible operators, then the operator $I - Z_0 H_0^{-1}(\lambda)$ will be invertible and therefore

$$H^{-1}(\lambda) = (I - Z_0 H_0^{-1}(\lambda))^{-1} H_0^{-1}(\lambda),\ \lambda \in (\lambda_0 - \varepsilon,\ \lambda_0 + \varepsilon).$$

Consider the function $f_0 \in L_1(\mathbf{R})$ such that $\widehat{f_0}(\lambda_0) \neq 0$ and $\operatorname{supp} \widehat{f_0} \subset (\lambda_0 - 2\varepsilon, \lambda_0 + 2\varepsilon)$ and $\widehat{f_0} = 1$ on $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and the operator-valued function

$$\widehat{f_0}(\lambda)\ H^{-1}(\lambda) = \widehat{f_0}(\lambda)\ (I - Z_0 H_0^{-1}(\lambda))^{-1} H_0(\lambda) = \widehat{\Phi}(\lambda),$$

the support of which lies in the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for $L\varphi = \Psi$, then from Lemma 3.2 it follows that $f_0 * \varphi \in D(L)$ and the following equality holds

$$L(f_0 * \varphi) = f_0 * \Psi \in B(\mathbf{R}; X),$$

i.e. $f_0 * \varphi$ is the solution of this equation.

Let us prove that the function $f_0 * \varphi$ can be represented in the form $\Phi * f_0 * \Psi$, where $\Phi : \mathbf{R} \to L(X)$ is summable operator function, the Fourier transform of which coincides with $\widehat{\Phi}(\lambda) = \widehat{f_0}(\lambda)\ H^{-1}(\lambda)$, $\lambda \in \mathbf{R}$ (from the kind of the function Φ it follows that it is the Fourier transform of the summable function φ as a function with compact support being the product of three functions, each of them is the Fourier transform of some summable function). The equality $f_0 * \varphi = \Phi * f_0 * \Psi$ will be proved if we determine that for each function $\Psi_0 \in B(\mathbf{R}; X)$ with $S(\psi_0) \subset (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ each solution of the condition $(C\Psi = \Psi_0)$ is representable in the kind $\Phi * \Psi_0$. By Lemma 3.3 there exists the sequence (φ_n) from $D(L)$ with properties: $\varphi_n \xrightarrow{c} \varphi$ and $L\varphi_n \to \Psi_0$, where φ_n aer trigonometrical polynomials. Therefore it is sufficient to verify the equality $\varphi_0 = \Phi * \Psi_0$ only for the trigonometrical polynomials of the type:

$$\varphi_0(t) = x_0 e^{i\mu_0 t}, \ \mu \in (\lambda_0 - \varepsilon, \ \lambda_0 + \varepsilon).$$

But for

$$L\varphi_0 \ (t) = (-A\mu_0^2 + i\mu_0 B + C) x_0 e^{i\mu_0 t} = H(\mu_0) x_0 e^{i\mu_0 t} = \Psi_0(t) = y_0 e^{i\mu_0 t}$$

and therefore

$$\varphi_0(t) = x_0 e^{i\mu_0 t} = H_0^{-1}(\mu) y_0 e^{i\mu_0 t}.$$

On the other hand

$$(\Phi * \Psi_0) \ (t) = \int_{-\infty}^{\infty} \Phi(t-s) y_0 e^{i\mu_0 t} ds = \widehat{\Phi}(\mu_0) y_0 e^{i\mu_0 t} =$$

$$= \widehat{f}_0(\mu_0) H_0^{-1}(\mu_0) y_0 e^{i\mu_0 t} = H_0^{-1}(\mu_0) y_0 e^{i\mu_0 t}.$$

From this follows the equality $\varphi_0 = \Phi * \Psi_0$ and the equality $f * \varphi = \Phi * (f_0 * \Psi)$ for $\Phi * (f_0 * \Psi) \in B(\mathbf{R}; X)$ and $\widehat{f}(\lambda_0) = 1 \neq 0$, then $\lambda_0 \overline{\in} S_0(\varphi)$, so $S_0(\varphi) \subset S(H)$. If $\lambda_0 \in S(H) \cup S(\Psi)$ then we select a function $f_0 \in L_1(\mathbf{R})$ such that $\widehat{f}_0(\lambda_0) \neq 0$ and $(\mathrm{Supp}\,\widehat{f}_0) \cap (S(H) \cap S(\Psi)) = \phi$. As we proved, $f_0 * \varphi = \Phi * f_0 * \Psi = 0$ for $f_0 * \Psi = 0$ (by property 3 of Lemma 2.1). Since $\widehat{f}_0(\lambda_0) \neq 0$, $\lambda_0 \overline{\in} S(\varphi)$, $S(\varphi) \subset S(H) \cup S(\Psi)$.

The following theorem follows immediately from Theorem 3.2 and 3.3.

Theorem 3.4. Each (generalized) bounded solution φ of the equation (3.1) is almost periodical, if the singular set of the bundle $H(\lambda)$ is less than countable and one of the following conditions holds:

1) The Banach space X does not contain the subspace isomorphic to the space C_0 of number sequences convergent to 0 (for instance, if X is reflexive or weakly complete one);

2) The solution φ has weakly compact domain of values;

3) the set $S(H) \cup S(\Psi)$ has no limit points on finite intervals of $\mathbf{R}$,

Proof. According to Theorem 3.3 $S_0(\varphi) \subset S(H)$ and $S(\varphi) \subset S(H) \cup S(\Psi)$. Taking into acount the assumtions of this theorem let us apply Theorem 3.2. To that end note that the function φ is uniformly continuous. It follows from the representation of φ as $\varphi = G * f_0$, where G is the constructed summable function. Actually

$$\| \varphi(t+\tau)-\varphi(t) \| = \left\| \int_{-\infty}^{\infty} [G(t+\tau-s)-G(t-s)]\, f_0(s)\, ds \right\| \leq$$

$$\leq \int_{-\infty}^{\infty} \| G_\tau(s) \| \, \| f_0(s) \| \, ds \to 0 \quad \tau \to \infty$$

uniformly on $t \in \mathbf{R}$ (the continuousness of the shift for summable operator functions).

Corollary 3.1. Let $A=I$, $B=0$ and the resolvent of the operator $R(\mu ; C)$ admits the estimation:

$$\| R(\mu, C) \| \leq \frac{\text{const}}{|Re\, \mu|}, \quad \mu \in \mathbf{C}, \ Re\, \mu \geq \omega_0 \in \mathbf{R}.$$

Then any generalized solution of the equation (3.1) is almost periodical if the spectrum $\sigma(C)$ in intersection with $R_+ = \{t \geq 0\}$ is less than countable (since the distance of C has compact resolvent and one of the conditions of Theorem 3.4 holds).

The results of Corollary 3.1 were obtained by Piskaryov [7] in the case that C is the generating operator of strongly continuous cosine function. The verification of the condition, when C is the generating operator of strongly cosine function is extremely complex (see [8]).

§4. The existence of the bounded solutions

Now consider the problem of the existence of the generalized bounded solutions of the equation (3.1) with continuous bounded (non-necessary almost periodical) function $\Psi : \mathbf{R} \to X$. As previously we suppose that the bundle

$$H(\lambda) = -A\lambda^2 + B(i\lambda) + C, \ \lambda \in \mathbf{R}$$

sasitsfies the conditions (1.1) and (1.2).

Lemma 4.1. Let $\Psi \in C(\mathbb{R} ; X)$ be the function with compact Börling spectrum $S(\Psi)$. Then the function Ψ is the bounding on $\mathbf{R}$ of some entire function of exponential type.

Such a result can be count familiar, for the Fourier transform of the function Ψ, considering that the generalized function has compact support which is equal to $S(\varphi)$. Therefore it is sufficient to use the analogs of the Viner-Pely theorem [9].

Lemma 4.2. If $\varphi \in D(L)$ then the following condition holds:

$$S(L\varphi) \subset S(\varphi).$$

Proof. Suppose that $\lambda_0 \overline{\in} S(\varphi)$. Then there exists a function $f \in L_1(\mathbf{R})$ such that $\widehat{f}(\lambda_0) \neq 0$ and

$$\operatorname{Supp} \widehat{f} \cap S(\varphi) = \phi$$

(which implies the equality $f * \varphi = 0$). From Lemma 3.2 follows that $f * \varphi \in D(L)$ and $f * L\varphi = L(f * \varphi) = 0$, i.e. $\lambda_0 \in S(L\varphi)$, $S(L\varphi) \subset S(\varphi)$.

Theorem 4.1. Let $S(H) \cap S(\Psi) = \phi$ and the set $S(\Psi)$ is compact. Then the equation (3.1) has generalized bounded solution $\varphi : \mathbf{R} \to X$ which (as a Ψ) is the restriction on $\mathbf{R}$ of some entire function of exponential type (moreover $S(\varphi) \subset S(\Psi)$).

Proof. Let $S(\Psi)$ be a compact set which by the assumption does not intersect with $S(H)$. Consider the open set V containing $S(\Psi)$ and not intersecting with $S(H)$ (and separated from it: $\operatorname{dist}(S(H), S(\Psi)) > 0$). One can point out the finite number of intervals covering $S(\Psi)$ and not intersecting with $S(H)$ and the function $f \in L_1(\mathbf{R})$ such that $\widehat{f}(\lambda) = 1$ in the neighbourhood V of the set $S(\Psi)$ and $\operatorname{supp} \widehat{f} \cap S(H) = \phi$. Consider now the function:

$$\Phi_1(\lambda) = \begin{cases} \widehat{f}(\lambda)\, H^{-1}(\lambda), & \lambda \in \operatorname{Supp} \widehat{f}(\lambda) \supset S(\varphi), \\ 0, & \lambda \overline{\in} \operatorname{Supp} \widehat{f}(\lambda). \end{cases}$$

The function Φ_1 is the Fourier transform of some summable operator function (because $H^{-1}(\lambda)$ locally belongs to the algebra of the Fourier transforms of the summable operator functions)

$$G_1(t) \in L_1(X), \ t \in \mathbf{R}.$$

From Note 1.2 it follows that the function of the type $x_0 e^{i\lambda_0 t}$ belongs to $D(L)$ and the vector x_0 belongs to $D(H(\lambda_0))$. If $\lambda_0 \in V$ and $x_0 \in D(H(\lambda_0))$ then

$$L(x_0 e^{i\lambda_0 t}) = H(\lambda_0)\, x_0\, e^{i\lambda_0 t}.$$

Since $H^{-1}(\lambda_0)$ exists, then

$$\varphi_0(t) = x_0 e^{i\lambda_0 t} = H^{-1}(\lambda_0)\, L(x_0 e^{i\lambda_0 t}) = H^{-1}(\lambda_0)\, \Psi_0(t), \ \Psi_0(t) = L(x_0 e^{i\lambda_0 t}).$$

Similar equality also holds for trigonometrical polynomials with spectrum from the set $V \subset \mathbb{R} \setminus S(H)$. So, since

$$\Phi_1(\lambda) = \widehat{f}(\lambda)\, H^{-1}(\lambda) = H^{-1}(\lambda), \ \lambda \in V,$$

then $\varphi_0 = \Phi * \Psi_0$ for each trigonometrical polynomial Ψ_0 with spectrum from V and $L\varphi_0 = \Psi_0$.

Now use the c-density of trigonometrical polynomials (see Lemma 3.3) in $D(L)$ and prove that the equation has the generalized solution φ of the type $\varphi=\Phi_1 * \Psi$. Actually, if (Ψ_n), $n>1$ is the sequence of periodical functions c-convergent to Ψ with spectrum from V, then $\varphi_n = \Phi_1 * \Psi_n \in D(L)$ e-converges to $\Phi_1 * \Psi$ and $L\varphi_n = \Psi_n$. From c-closedness of the operator L it follows that $L\varphi = \Psi$. Moreover, it is proved that $\varphi = \Phi_1 * \Psi$ from which, for instance, implies that $S(\varphi) \subset S(\psi)$ (see the statement 2 of Lemma 2.1).

From the proof of Theorem 4.2 follows the uniqueness of the solution (I), the spectrum of which does not intersect with $S(H)$. Actually we proved the local invertibility of the operator L on the subspaces of functions, the spectrum of which is compact and does not intersect.

Theorem 4.2. The equation (3.1) has the unique solution for each function $\psi \in C(\mathbb{R}; X)$ if $S(H) = \phi$.

Proof. Consider the arbitrary point $\lambda_0 \in \mathbb{R}$ and vector $x_0 \in X$. Then the equation $Lx = y_0 e^{i\lambda_0 t}$ has the unique solution $a_0(t)$, i.e. $(La_0)(t) = y_0 e^{i\lambda_0 t}$. Consider $\lambda_1 \neq \lambda_0$ and the function $\hat{f} \in L_1(\mathbb{R})$ with properties $\hat{f}(\lambda_1) \neq 0$, $\hat{f}(\lambda_0) = 0$. Then from Lemma 3.2 it follows that $f * a_0 \in D(L)$ and $L(f * a_0) = f * (y_0 e^{i\lambda_0 t}) = 0$. Then $L(f * a_0) = 0$ and therefore $f * a_0 = 0$. It means that $\lambda_0 \in S(a_0)$, i.e. $S(a_0) = \{\lambda_0\}$. As is known in this case a_0 has the type $a_0(t) = x_0 e^{i\lambda_0 t}$ and $H(\lambda_0)x_0 = y_0$; therefore for each vector $y_0 \in X$ there exists the unique vector $X_0 \in D(H(\lambda_0))$.

From the closedness of the operator $H(\lambda_0)$ (since $H_0(\lambda_0) = H(\lambda_0) + Z_0 I$ is invertible operator) follows its invertibility, i.e. $\lambda_0 \in S(H)$. From this by arbitrarity of λ_0 we obtain that $S(H) = \phi$.

Sufficiency: Let $S(H) = \phi$. Consider the operator function $H_0(\lambda) = H(\lambda) + Z_0 I$, $\lambda \in \mathbb{R}$. Remember that for this function hold conditions (1.1), (1.2) from which it follows that the function $H_0^{-1}(\lambda_0) \in L(X)$, $\lambda \in \mathbb{R}$ is representable in the form $H_0^{-1}(\lambda) = \hat{\Phi}(\lambda)$, where Φ is summable operator function $H(\lambda)$. Represent the function $H(\lambda)$ in the form

$$H(\lambda) = H_0(\lambda) - Z_0 I = H_0(\lambda)\,(I - Z_0 H_0^{-1}(\lambda)).$$

Since $H(\lambda)$ is invertible, the operators $I - Z_0 H_0^{-1}(\lambda)$ are invertible for each $\lambda \in \mathbb{R}$. Therefore the function $H^{-1}(\lambda)$ is representable in the form

$$H^{-1}(\lambda) = H_0^{-1}(\lambda)\,(I - Z_0 H_0^{-1}(\lambda))^{-1}$$

of product of two operator functions $H_0^{-1}(\lambda)$ and $(I - Z_0 H_0^{-1}(\lambda))^{-1}$, the first of which is the Fourier transform of summable function by the theorem of

Bochner-Philips [10]. Then the function $H^{-1}(\lambda)$ is the Fourier transform of some summable function $\Phi_1(t) \in L(X)$, $t \in \mathbf{R}$. Therefore the solution φ of the equation (3.1) with $\psi(t) = y_0 e^{i\lambda_0 t}$ admits the representation of the form

$$\varphi(t) = H^{-1}(\lambda_0)\, y_0\, e^{i\lambda_0 t} = \Phi_1 * \psi_0 .$$

From this the equality $\varphi_0 = \Phi_1 * \psi$ can be extended immediately on almost-periodical function ψ (making use of the theorem of approximation for almost periodical functions). So, the operator L is invertible on the subspace of almost periodical functions and

$$L^{-1}\psi = \Phi_1 * \psi, \quad \psi \in B(\mathbf{R}\,;X).$$

In order to extend this formula on functions ψ from $C(\mathbf{R}\,;X)$ it is sufficient to use Lemma 3.3.

Note 4.1. For the equation of the first order $L_0 x = \dot{x} - Ax = \psi(t)$, $\psi \in C(\mathbf{R}\,;X)$ under the condition that A -infinitesimal generating operator of strongly continuous semigroup of operators, the result analogous to Theorem 4.2 has not jet been obtained. But, X is Hilbert space and the spectrum $\sigma(A)$ of the operator intersects with imaginary axis $i\,\mathbf{R}=\{i\,t : t \in \mathbf{R}\}$ (which respects to our condition $S(H) = \phi$) under the condition $\sup_{t\in\mathbf{R}} \|(i\,\lambda\, I - A)^{-1}\| < \infty$ substituting our conditions (1.1) and (1.2). By immediate combination of the proofs of Theorems 4.1 and 4.2 one can prove the following theorem:

Theorem 4.3. Suppose that the singular set $S(H)$ of the bundle $H(\lambda)$, $\lambda \in \mathbf{R}$ is compact and the following condition holds:

$$S(H) \cap S(\psi) = \phi.$$

Then the equation (3.1) has the unique bounded generalized solution $\varphi \in C(\mathbf{R}\,;X)$ having the property $S(\varphi) \subset S(\psi)$.

Note 4.2. We can attach various values to the notion of the classical solution of the equation (3.1). For instance, the classical solution φ of the equation (3.4.) can be defined as twicely continuous differentiable function, the values of which and of which first two derivatives lie in $D(A) \cap D(B) \cap D(C)$ and for which the equality holds

$$A\ddot{\varphi} + B\dot{\varphi} + C\varphi = \psi.$$

If $x_0\, e^{i\lambda_0 t} = \varphi_0(t)$ is the function with $x_0 \in D(A) \cap D(B) \cap D(C)$ then it is clear that

$$L(x_0\, e^{i\lambda_0 t}) = H_0(\lambda_0)\, x_0\, e^{i\lambda_0 t} = \psi_0(t)$$

and very simple analysis suggests that φ_0 is the solution of the equation $L\varphi_0 = \psi_0$ in general sense. It is clear that from this considerations it follows also that the similar equality can be extended on trigonometrical polynomials and then by Lemma 3.3 on each classical solution φ of the equation $L\varphi = \psi$.

REFERENCES

[1] RAGIMOV, M.B. : *Operator pencils and almost periodicity of bounded solutions of second-order differential equations*, Dokl. Akad. Nauk SSSR, **314** (1990), N. 5, English transl. in Soviet Math. Dokl., **42** (1991), N. 2, 608-611.

[2] CUCHI, S. : *On abstract Cauchy problems in the sense of hyper-functions*, Lecture Notes in Math., V. **287** (1973), 135-152.

[3] FATTORINI, H.O. : *Ordinary differential equations on linear topological spaces I*, J. Differential Equations, **5** (1969), No. 1, 72-105.

[4] MEL'NIKOVA, I.V. and FILINKOV. A.I. : Dokl. Akad. Nauk SSSR, **300** (1988), 280-284.

[5] LOOMIS, L.H. : *Spectral characterization of almost periodic functions*, Ann. Math., 72 (1960), No. 2, 362-368.

[6] BASKAKOV, A.G. : Mat. Zametki, 24 (1978), 195-206.

[7] PISKARYOV, S.I. : Mat. Sb., **118** **(160)** (1982), 386-398.

[8] KOSTIN, V.A. : Dokl. Akad. Nauk. SSSR, **307** (1989), N. 4, 386-389.

[9] IOSIDA, K. : Functional Analysis, M. Mir, 1967.

[10] BOCHNER, S. and PHILLIPS, R.S. : *Absolute convergent Fourier expansions for non-commutative normed rings*, Ann. Math., 43 (1942), N. 2, 409-418.