ON P-SASAKIAN MANIFOLD

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Summary : In this paper, the nature of the P-Sasakian manifold has been studied, when $R(X, Y) . \bar{C} = 0$, where $\bar{C}$ is the concircular curvature tensor.

P-SASAKİAN MANİFOLD HAKKINDA

Özet : Bu çalışmada, $\bar{C}$ konsirküler eğrilik tensörü olmak üzere, $R(X, Y) . \bar{C} = 0$ olması durumunda P-Sasakian manifoldun yapısı incelenmektedir.

Introduction. Let (M^n, g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the conditions

1) $(\nabla_X \eta) Y - (\nabla_Y \eta) X = 0,$

2) $(\nabla_X \nabla_Y \eta) Z = - g(X, Z) \eta(Y) - g(X, Y) \eta(Z) + 2\eta(X) \eta(Y) \eta(Z)$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M^n, g) admits a vector field ξ and a (1-1) tensor field ϕ such that

3) $g(X, \xi) = \eta(X),$

4) $\eta(\xi) = 1,$

5) $\nabla_X \xi = \phi X$

then such a manifold is called a Para-Sasakian manifold or briefly a P-Sasakian manifold ([2], [3]). The present paper deals with a type of P-Sasakian manifold in which $R(X, Y) . \bar{C} = 0$, where $\bar{C}$ is the concircular curvature tensor, R is the curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. It is shown that such a manifold is locally isometric with a unit sphere $S^n(1)$. Further, it is shown that a concircularly symmetric P-Sasakian manifold is concircularly flat.

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1. **Preliminaries.** Let S and r denote respectively the Ricci tensor of type (0,2) and the scalar curvature of M^n. It is known that in a P-Sasakian manifold the following relations hold ([1], [2], [3]):

$$\phi\,\xi = 0 \tag{1.1}$$

$$\phi^2 X = X - \eta(X)\,\xi \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \tag{1.3}$$

$$S(X, \xi) = -(n-1)\,\eta(X) \tag{1.4}$$

$$\eta(R(X, Y)Z) = g(X, Z)\,\eta(Y) - g(Y, Z)\,\eta(X) \tag{1.5}$$

$$R(\xi, X)\,Y = \eta(Y)\,X - g(X, Y)\,\xi \tag{1.6}$$

$$R(\xi, X)\,\xi = X - \eta(X)\,\xi \tag{1.7}$$

$$R(X, Y)\,\xi = \eta(X)\,Y - \eta(Y)\,X \tag{1.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\,\eta(X)\,\eta(Y). \tag{1.9}$$

The above results will be used in the next section.

2. **P-Sasakian manifold satisfying $R(X, Y)\,.\,\bar{C}=0$.** The concircular curvature tensor, denoted by $\bar{C}$, is defined as

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{2.1}$$

Let

$$R(X, Y)\,.\,\bar{C} = 0.$$

Then one must have

$$R(X, Y)\,\bar{C}(U, V)W - \bar{C}(R(X, Y)U, V)W - \\ - \bar{C}(U, R(X, Y)V)W - \bar{C}(U, V)R(X, Y)W = 0$$

or

$$g(R(\xi, Y)\,\bar{C}(U, V)W, \xi) - g(\bar{C}(R(\xi, Y)U, V)W, \xi) - \\ - g(\bar{C}(U, R(\xi, Y)V)W, \xi) - g(\bar{C}(U, V)R(\xi, Y)W, \xi) = 0. \tag{2.2}$$

Taking $X = \xi$ in (2.1) and using (1.6) one gets

$$\bar{C}(\xi, Y)Z = \left\{1 + \frac{r}{n(n-1)}\right\}\{\eta(Z)\,Y - g(Y, Z)\,\xi\}. \tag{2.3}$$

Also, from (2.3), one finds

$$\eta(\bar{C}(\xi, Y)Z) = \left\{1 + \frac{r}{n(n-1)}\right\}\{\eta(Z)\,\eta(Y) - g(Y, Z)\}. \tag{2.4}$$

Taking $X = Z = U$ in (2.1) and using (1.5) we find

$$\eta(\bar{C}(U, V) U) = \left\{1 + \frac{r}{n(n-1)}\right\} \{g(U, V)\eta(V) - g(V, U)\eta(U)\}. \tag{2.5}$$

Using (1.6) in (2.2) we get

$$\begin{aligned} &\eta(Y)\eta(\bar{C}(U, V) W) - g(Y, \bar{C}(U, V) W) - \\ &- \eta(U)\eta(\bar{C}(Y, V) W) + g(Y, U)\eta(\bar{C}(\xi, V) W) - \\ &- \eta(V)\eta(\bar{C}(U, Y) W) + g(Y, V)\eta(\bar{C}(U, \xi) W) - \\ &- \eta(W)\eta(\bar{C}(U, V) Y) + g(Y, W)\eta(\bar{C}(U, V)\xi) = 0. \end{aligned} \tag{2.6}$$

Taking $Y = U$ in (2.6), we obtain

$$\begin{aligned} &g(U, \bar{C}(U, V) W) - g(U, V)\eta(\bar{C}(\xi, V) W) + \\ &+ \eta(V)\eta(\bar{C}(U, U) W) - g(U, V)\eta(\bar{C}(U, \xi) W) + \\ &+ \eta(W)\eta(\bar{C}(U, V) U) - g(U, W)\eta(\bar{C}(U, V)\xi) = 0. \end{aligned} \tag{2.7}$$

Using (2.4) and (2.5), one gets from (2.7)

$$\begin{aligned} g(U, \bar{C}(U, V) W) + \left\{1 + \frac{r}{n(n-1)}\right\} \{g(U, U) g(V, W) - \\ - g(U, V) g(U, W)\} = 0. \end{aligned} \tag{2.8}$$

Using (2.1), it follows from (2.8) that

$$g(U, R(U, V) W) = g(U, V) g(U, W) - g(U, U) g(V, W). \tag{2.9}$$

Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then the sum for $u = e_i$, $1 \leq i \leq n$ gives

$$S(V, W) = -(n-1) g(V, W). \tag{2.10}$$

Using (2.1) and (2.4), we get from (2.6)

$$\begin{aligned} &\eta(Y)\eta(R(U, V) W) - g(Y, R(U, V) W) - \eta(U)\eta(R(Y, V) W) + \\ &+ g(Y, U)\eta(W)\eta(V) - g(Y, U) g(V, W) - \eta(V)\eta(R(U, Y) W) + \\ &+ g(Y, V) g(U, W) - g(Y, V)\eta(W)\eta(U) - \eta(W)\eta(R(U, V) Y) = 0. \end{aligned} \tag{2.11}$$

Using (1.5), one gets from (2.11)

$$g(Y, R(U, V) W) = g(U, W) g(Y, V) - g(Y, U) g(V, W) \tag{2.12}$$

whence

$$R(U, V) W = g(U, W) V - g(V, W) U. \tag{2.13}$$

From (2.10) one finds

$$r = -n(n-1). \tag{2.14}$$

Also, on using (2.14), we find from (2.1)

$$\bar{C}(X, Y) Z = R(X, Y) Z + g(Y, Z) X - g(X, Z) Y. \tag{2.15}$$

Consequently one finds from above

$$\bar{C}(\xi, X) \xi = 0. \tag{2.16}$$

Thus we state

Theorem 1. Let (M^n, g) be a P-Sasakian manifold with $R(X, Y) . \bar{C} = 0$. Then

i) M is an Einstein manifold with scalar curvature $-n(n-1)$

ii) M is of constant curvature

iii) $\bar{C}(\xi, X) \xi = 0$ for every X.

3. Concircularly symmetric P-Sasakian Manifold. For a concircularly symmetric P-Sasakian manifold, $\nabla \bar{C}=0$. Hence for such a manifold $R(X, Y) . \bar{C}=0$. Thus (2.13) holds and consequently from (2.15) we have

$$\bar{C}(X, Y) Z = 0.$$

Thus we state the following theorems :

Theorem 2. A concircularly symmetric P-Sasakian manifold is locally isometric with a unit sphere $S^n(1)$.

Theorem 3. A concircularly symmetric P-Sasakian manifold is concircularly flat.

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