ON THE BOUNDEDNESS AND THE STABILITY RESULTS FOR THE SOLUTIONS OF CERTAIN FIFTH ORDER DIFFERENTIAL EQUATIONS

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Summary : The paper investigates the equation (1.1) in the two cases : (i) $p \equiv 0$, (ii) $p\,(\not\equiv 0)$ satisfies $|p(t, x, y, z, w, u)| \leq (1+|y|+|z|+|w|+|u|)\, q(t)$, where $q(t)$ is a nonnegative function of t. For the case (i) the asymptotic stability in the large of the trivial solution $x = 0$ is investigated and for the case (ii) a boundedness result is obtained for solutions of (1.1). The results obtained here extend several well-known results.

5. MERTEBEDEN BELİRLİ DİFERANSİYEL DENKLEMLERİN ÇÖZÜMLERİ İÇİN SINIRLILIK VE STABİLİTE SONUÇLARI HAKKINDA

Özet : Bu çalışmada (1.1) denklemi şu iki halde incelenmektedir: (i) $p \equiv 0$ dır, (ii) $p\,(\not\equiv 0)$, $|p(t, x, y, z, w, u)| \leq (1+|y|+|z|+|w|+|u|)\, q(t)$ bağıntısını gerçekler ki, burada $q(t)$, t nin negatif olmayan bir fonksiyonudur. (i) hali için $x=0$ trivial çözümünün asimtotik stabilitesi incelenmekte, (ii) halinde de (1.1) in çözümleri için bir sınırlılık sonucu elde edilmektedir ve burada elde edilen sonuçlar, bilinen bazı sonuçlara genişletilmektedir.

1. Introduction and statement of the results

We shall consider the non-linear fifth order differential equation

$$x^{(5)}+\varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\, x^{(4)}+\psi(\ddot{x}, \dddot{x})+h(\ddot{x})+g(\dot{x})+f(x)=p(t, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \qquad (1.1)$$

in which the functions φ, ψ, h, g, f and p, which depend only on the arguments shown explicitly, are such that $\varphi(x, y, z, w, u)$, $\psi(z, w)$, $\frac{\partial}{\partial z}\psi(z, w)$, $g'(y)$, $f'(x)$ and $p(x, y, z, w, u)$ are continuous for all values of t, x, y, z, w and u.

The boundedness and stability properties of solutions of non-linear fourth order differential equations have been the subject of intensive investigation. Many of these results are summarized in [4]. Similar investigations have been carried out on various special cases of (1.1) by a number of authors.

Key words : Nonlinear differential equations of the fifth order, stability, boundedness, V-function.

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Chukwu [2] dealt with the equation of the form

$$x^{(5)} + ax^{(4)} + f_2(\dddot{x}) + c\ddot{x} + f_4(\dot{x}) + f_5(x) = 0 \tag{1.2}$$

and presented sufficient conditions for asymptotic stability in the large of the zero solution for that equation.

A similar result was also obtained for the equation

$$x^{(5)} + f_1(\dddot{x})\, x^{(4)} + f_2(\dddot{x}) + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = 0 \tag{1.3}$$

by Abou-El-Ela and Sadek [1].

Furthermore, recently, in [5], Yuanhong studied fifth order non-linear differential equations of the form

$$x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\, x^{(4)} + b\dddot{x} + h(\ddot{x}) + g(\dot{x}) + f(x) = p(t). \tag{1.4}$$

He obtained some results concerning asymptotic stability in the large of the zero solution of (1.4) with $p(t) \equiv 0$ and the boundedness of solutions of (1.4) with $p(t) \neq 0$.

The assumptions which will be established here are generalizations of the Routh-Hurwitz conditions

$$\begin{gathered} a > 0,\ ab - c > 0,\ (ab - c)\, c - (ad - e)\, a > 0, \\ \Delta = (cd - be)\,(ab - c) - (ad - e)^2 > 0,\ e > 0, \end{gathered} \tag{1.5}$$

which are necessary and sufficient for the asymptotic stability in the large of the trivial solution of the linear differential equation

$$x^{(5)} + ax^{(4)} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = 0 \tag{1.6}$$

with constant coefficients.

Equation (1.1) has an equivalent system

$$\begin{gathered} \dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = u, \\ \dot{u} = -\varphi(x, y, z, w, u)\, u - \psi(z, w) - h(z) - g(y) - f(x) + p(t, x, y, z, w, u). \end{gathered} \tag{1.7}$$

We start with the case $p \equiv 0$ in (1.1) and prove here that:

Theorem 1. In addition to the fundamental assumptions of φ, ψ, h, g and f, we suppose that:

I) The constants a, b, c, d and e satisfy (1.5) and following two inequalities:

$$\Delta_1 = \frac{(cd - be)\,(ab - c)}{ad - e} - (ag'(y) - e) > 2\varepsilon b \text{ for all } y, \tag{1.8}$$

$$\Delta_2 = \frac{cd - be}{ad - e} - \frac{d'(ad - e)}{d(ab - e)} - \frac{\varepsilon}{a} > 0 \text{ for all } y, \tag{1.9}$$

where

$$d' = \begin{cases} \dfrac{g(y)}{y}, & y \neq 0, \\ g'(0), & y = 0. \end{cases} \tag{1.10}$$

II) $f(0) = g(0) = h(0) = \psi(z, 0) = 0$,

$\dfrac{f(x)}{x} \geq \alpha > 0$ for all $x \neq 0$, where α is a positive constant,

$\dfrac{g(y)}{y} \geq d$ for all $y \neq 0$,

$\dfrac{h(z)}{z} \geq c$ for all $z \neq 0$,

$\dfrac{\psi(z, w)}{w} \geq b$ for all z and $w \neq 0$,

$\varphi(x, y, z, w, u) \geq a$ for all x, y, z, w and u.

III) $f'(x) \leq e$ for all x,

$$(f'(x) - e)^2 < \min\left[\frac{\varepsilon^2 d}{16}, \frac{\varepsilon\Delta_1 d}{32a^2}\right] \text{ for all } x. \tag{1.11}$$

IV) $g'(y) - \dfrac{g(y)}{y} \leq \beta$ for all $y \neq 0$, $\qquad$ (1.12)

where β is a positive constant such that

$$\beta < \frac{e\Delta}{d^2(ab - c)}, \tag{1.13}$$

$$[g'(y) - d]^2 < \frac{\varepsilon\Delta_1}{64} \text{ for all } y. \tag{1.14}$$

V) $\left[\dfrac{h(z)}{z} - c\right]^2 < \dfrac{\varepsilon d \Delta_1}{16\,\delta^2}$ for all $z \neq 0$, $\qquad$ (1.15)

where δ is a positive constant satisfying

$$\delta = \frac{e(ab - c)}{ad - e} + \varepsilon, \tag{1.16}$$

$$\left[\frac{h(z)}{z} - c\right]^2 \leq \frac{\Delta_1}{16}(\varphi - a) \text{ for all } x, y, w, u \text{ and } z \neq 0, \tag{1.17}$$

and

$$\varphi - a < \varepsilon_0 = \min\left[\frac{\varepsilon}{4a^2}, \frac{\varepsilon d}{4\delta^2}, \frac{\Delta_1 (ad-e)^2}{16d^2 (ab-c)^2}\right] \text{ for all } x, y, z, w \text{ and } u. \quad (1.18)$$

VI) $$\left[\frac{\psi(z, w)}{w} - b\right]^2 < \min\left[\frac{\varepsilon\Delta_1 (ad-e)^2}{64d^2 (ab-c)^2}, \frac{\varepsilon^2 d}{16\delta^2}\right] \text{ for all } z \text{ and } w \neq 0, \quad (1.19)$$

and

$$\psi_z (z, w) \leq 0 \text{ for all } z, w. \quad (1.20)$$

Then every solution $(x(t), y(t), z(t), w(t), u(t))$ of system (1.7) satisfies

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) + u^2(t) \to 0 \text{ as } t \to \infty, \quad (1.21)$$

provided that the positive constants ε and ε_0 are sufficiently small.

For the case $p(t, x, y, z, w, u) \neq 0$, we shall prove

Theorem 2. Suppose that

(I) conditions (I)-(VI) of Theorem 1 hold,

(II) the function $p(t, x, y, z, w, u)$ satisfies $|p(t, x, y, z, w, u)| \leq (1 + |y| + |z| + |w| + |u|)\, q(t)$, where $q(t)$ is a nonnegative and continuous function of t, and satisfies $\int_0^t q(s)\, ds \leq A < \infty$, for all $t \geq 0$, A is a positive constant. Then for any given finite x_0, y_0, z_0, w_0, u_0 there exists a constant $D = D(x_0, y_0, z_0, w_0, u_0)$, such that any solution $(x(t), y(t), z(t), w(t), u(t))$ of system (1.7) determined by

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0,\ w(0) = w_0,\ u(0) = u_0,$$

satisfies for all $t \geq 0$,

$$|x(t)| \leq D,\ |y(t)| \leq D,\ |z(t)| \leq D,\ |w(t)| \leq D,\ |u(t)| \leq D. \quad (1.22)$$

Remark 1. When $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = a$, $\psi(\ddot{x}, \dddot{x}) = b\dddot{x}$, $h(\ddot{x}) = c\ddot{x}$, $g(\dot{x}) = d\dot{x}$ and $f(x) = ex$ and $p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = 0$, equation (1.1) reduces to the linear differential equation (1.6) with constant coefficients and conditions (I)-(VI) of Theorem 1 reduce to the corresponding conditions of Routh-Hurwitz criterion.

Remark 2. When $\psi(\ddot{x}, \dddot{x}) = b\dddot{x}$, $p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = p(t)$, then the conditions of Theorem 1 and Theorem 2 reduce to those of Yuanhong [5].

2. The function $V(x, y, z, w, u)$

The proofs of the theorems depend on a scalar differentiable function $V(x, y, z, w, u)$. This function and its time derivative satisfy fundamental inequalities. The function $V = V(x, y, z, w, u)$ is defined by:

$$2V = u^2 + 2awu + \frac{2d(ab-c)}{ad-e} zu + 2\int_0^w \psi(z, s)\, ds + \left[a^2 - \frac{d(ab-c)}{ad-e}\right] w^2 +$$

$$+2\left[c + \frac{ad(ab-c)}{ad-e} - \delta\right] zw + 2\delta yu + 2a\delta yw + 2wf(x) + 2wg(y) +$$

$$+2a\int_0^z h(s)\, ds + \left[\frac{bd(ab-c)}{ad-e} - d - a\delta\right] z^2 + 2b\delta yz + 2azg(y) - 2eyz +$$

$$+2azf(x) + (\delta c - ae) y^2 + \frac{2d(ab-c)}{ad-e}\int_0^y g(s)\, ds + \frac{2d(ab-c)}{ad-e} yf(x) + \tag{2.1}$$

$$+2\int_0^x f(s)\, ds,$$

where $\varepsilon > 0$ and $\delta > 0$ are constants satisfying (1.8), (1.9) and (1.16).

The properties of the function V, which are required for the proof of (1.21) and (1.22), are summarized in Lemma 1 and Lemma 2.

Lemma 1. Under the conditions of Theorem 1, there exist positive constants $D_i \equiv D_i(a, b, c, d, e, \alpha, \beta, \varepsilon)$ $(i = 1, 2, 3, 4, 5)$ such that

$$V \geq D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 w^2 + D_5 u^2$$

for all x, y, z, w, u.

Proof. $V(0, 0, 0, 0,0) = 0$, since $f(0) = g(0) = h(0) = \psi(z, 0) = 0$.

Also, since $\psi(z, 0) = 0$ and $\dfrac{\psi(z, w)}{w} \geq b$ $(w \neq 0)$ it is clear that $\displaystyle\int_0^w \psi(z, s)\, ds \geq bw^2$.

Therefore (2.1) takes the form

$$2V \geq \left[u + aw + \frac{d(ab-c)}{ad-e} z + \delta y\right]^2 + \frac{d\Delta}{(ad-e)^2}\left[z + \frac{e}{d} y\right]^2 + \Delta_2 [w + az]^2 +$$

$$+\frac{d(ad-e)}{d'(ab-c)}\left[\left(\frac{ab-c}{ad-e}\right) f(x) + \left(\frac{ab-c}{ad-e}\right) d'y + \left(\frac{ad'}{d}\right) z + \left(\frac{d'}{d}\right) w\right]^2 +$$

$$+2\delta\int_0^x f(s)\, ds - \frac{d(ab-c)}{d'(ad-e)} f^2(x) + \frac{d(ab-c)}{ad-e}\left[2\int_0^y g(s)\, ds - yg(y)\right] +$$

$$+\left[\delta c-ae-\frac{e^2\Delta}{d(ad-e)^2}-\delta^2\right]y^2+a\left[2\int_0^z h(s)\,ds-cz^2\right]+$$

$$+\left(\frac{\varepsilon}{a}\right)w^2+2\varepsilon\left[\frac{cd-be}{ad-e}\right]yz, \tag{2.2}$$

where Δ and Δ_2 are defined by (1.5) and (1.9), respectively.

The terms on the right-hand side of the inequality (2.2) are the same as the terms on the right-hand side of the inequality (2.4) in [5, pp. 269, 270]. In fact, the estimation there for the terms on the right-hand side of (2.2) yields

$$V \geq D_1\, x^2 + D_2\, y^2 + D_3\, z^2 + D_4\, w^2 + D_5\, u^2. \tag{2.3}$$

This completes the proof of Lemma 1.

Lemma 2. Let all the conditions of Theorem 1 be satisfied. Then there exist positive constants $D_i \equiv D_i\,(b, d, \varepsilon)$ $(i = 6, 7, 8)$ such that every solution (x, y, z, w, u) of system (1.7) satisfies

$$\dot{V} \equiv \frac{d}{dt}\, V(x, y, z, w, u) \leq -\,(D_6\, y^2 + D_7\, z^2 + D_8\, w^2). \tag{2.4}$$

Proof. A straightforward calculation using the identity

$$\dot{V} = \frac{\partial V}{\partial x}\, y + \frac{\partial V}{\partial y}\, z + \frac{\partial V}{\partial z}\, w + \frac{\partial V}{\partial w}\, u + \frac{\partial V}{\partial u}\, \dot{u}$$

yields

$$\dot{V}=-(\varphi-a)u^2-\left[a\,\frac{\psi(z,w)}{w}-c+\delta-\frac{ad(ab-c)}{ad-e}\right]w^2-$$

$$-\left[\frac{d(ab-c)}{ad-e}\,\frac{h(z)}{z}-\{\delta b+(ag'(y)-e)\}\right]z^2-$$

$$-\left[\delta y g(y)-\frac{d(ab-c)}{ad-e}\,f'(x)y^2\right]-a(\varphi-a)wu-$$

$$-\left[\frac{h(z)}{z}-c\right]zu-\frac{d(ab-c)}{ad-e}\,(\varphi-a)zu-\delta(\varphi-a)yu+$$

$$+\,[g'(y)-d]zw+[f'(x)-e]\,yw-\delta\left[\frac{h(z)}{z}-c\right]yz-$$

$$-\,a[e-f'(x)]yz-\frac{d(ab-c)}{ad-c}\,\psi(z,w)z-$$

$$-\delta y\psi(z,w)+\frac{bd(ab-c)}{ad-e}\,zw+b\delta yw+w\int_0^w \psi_\varepsilon(z,s)ds. \tag{2.5}$$

It follows from $\frac{\psi(z,w)}{w} \geq b$ and (1.16) that

$$\left[a\,\frac{\psi(z,w)}{w} - c + \delta - \frac{ad(ab-c)}{ad-e}\right] w^2 \geq \left[ab - c + \delta - \frac{ad(ab-c)}{ad-e}\right] w^2 = \varepsilon w^2. \quad (2.6)$$

By using $\frac{h(z)}{z} \geq c$, (1.8) and $ab - c + \delta - \frac{ad(ab-c)}{ad-e} = \varepsilon$ we find

$$\frac{d(ab-c)}{ad-e}\,\frac{h(z)}{z} - \{\delta b + (ag'(y) - e)\} \geq$$

$$\geq \frac{(cd-be)(ab-c)}{ad-e} - (ag'(y) - e) - \varepsilon b >$$

$$> \frac{1}{2}\left[\frac{(cd-be)(ab-c)}{ad-e} - (ag'(y) - e)\right]. \quad (2.7)$$

From $\frac{g(y)}{y} \geq d$ and $f'(x) \leq e$, we obtain

$$-\left[\delta y g(y) - \frac{d(ab-c)}{ad-e} f'(x) y^2\right] \leq -\varepsilon d y^2 - \frac{ab-c}{ad-e}\,[de - df'(x)]\,y^2 \leq -\varepsilon d y^2. \quad (2.8)$$

Because of (1.20), it follows that

$$w \int_0^w \psi_z(z, s)\,ds \leq 0. \quad (2.9)$$

Combining inequalities (2.6)-(2.9) in (2.5) we get

$$\dot{V} \leq -\frac{\varepsilon d}{8} y^2 - \frac{\Delta_1}{8} z^2 - \frac{\varepsilon}{4} w^2 - \sum_{t=7}^{14} V_t, \quad (2.10)$$

where

$$V_7 = \left(\frac{1}{4}\right)(\varphi - a)u^2 + \left[\frac{h(z)}{z} - c\right] zu + \left(\frac{\Delta_1}{16}\right) z^2,$$

$$V_8 = \left(\frac{1}{4}\right)(\varphi - a)u^2 + \frac{d(ab-c)}{ad-e}(\varphi - a)\,zu + \left(\frac{\Delta_1}{16}\right) z^2,$$

$$V_9 = \left(\frac{1}{4}\right)(\varphi - a)\,u^2 + a\,(\varphi - a)\,wu + \left(\frac{\varepsilon}{4}\right) w^2,$$

$$V_{10} = \left(\frac{1}{4}\right)(\varphi - a)\,u^2 + \delta\,(\varphi - a)\,yu + \left(\frac{\varepsilon d}{4}\right) y^2,$$

$$V_{11} = \left(\frac{\varepsilon}{4}\right) w^2 - [g'(y) - d]\, zw + \frac{d(ab-c)}{ad-e}\left[\frac{\psi(z,w)}{w} - b\right] zw + \left(\frac{\Delta_1}{16}\right) z^2 ,$$

$$V_{12} = \left(\frac{\varepsilon}{4}\right) w^2 + [e - f'(x)]\, yw + \delta\left[\frac{\psi(z,w)}{w} - b\right] yw + \left(\frac{\varepsilon d}{4}\right) y^2 ,$$

$$V_{13} = \left(\frac{\Delta_1}{16}\right) z^2 + \delta\left[\frac{h(z)}{z} - c\right] yz + \left(\frac{\varepsilon d}{4}\right) y^2 ,$$

$$V_{14} = \left(\frac{\Delta_1}{16}\right) z^2 + a[e - f'(x)]\, yz + \left(\frac{\varepsilon d}{8}\right) y^2 .$$

The functions V_7, V_8, V_9, V_{10}, V_{13} and V_{14} are the components of $\dot{V}$ in the proof of [5, Lemma 2]. For precisely the same reasons as in [5]

$$V_7 \geq 0,\ V_8 \geq 0,\ V_9 \geq 0,\ V_{10} \geq 0,\ V_{13} \geq 0 \text{ and } V_{14} \geq 0.$$

Now consider the expressions

$$V_{11} = \left(\frac{\varepsilon}{4}\right) w^2 - [g'(y) - d]\, zw + \frac{d(ab-c)}{ad-e}\left[\frac{\psi(z,w)}{w} - b\right] zw + \left(\frac{\Delta_1}{16}\right) z^2$$

and

$$V_{12} = \left(\frac{\varepsilon}{4}\right) w^2 + [e - f'(x)]\, yw + \delta\left[\frac{\psi(z,w)}{w} - b\right] yw + \left(\frac{\varepsilon d}{4}\right) y^2 .$$

By similar estimation using (1.11), (1.14) and (1.19) we obtain $V_{11} \geq 0$ and $V_{12} \geq 0$. Summing up the above discussion we obtain

$$\dot{V} \leq -\left(\frac{\varepsilon d}{8} y^2 + \frac{\Delta_1}{8} z^2 + \frac{\varepsilon}{4} w^2\right). \tag{2.11}$$

By (1.8), inequality (2.11) implies that

$$\dot{V} \leq -\left(\frac{\varepsilon d}{8} y^2 + \frac{\varepsilon b}{4} z^2 + \frac{\varepsilon}{4} w^2\right), \tag{2.12}$$

which verifies (2.4).

Proof of Theorem 1. See [5].

Proof of Theorem 2. Consider the function V defined by (2.1). Then, since $p \neq 0$, under the conditions of Theorem 2, the conclusion of Lemma 2 can be revised as follows:

$$\dot{V}(x, y, z, w, u) \leq -D_6 y^2 - D_7 z^2 - D_8 w^2 + \\ + \left[u + aw + \frac{d(ab-c)}{ad-e} z + \delta y\right] p(t, x, y, z, w, u). \tag{2.13}$$

Let

$$D_9 = \max\left(1, a, \frac{d(ab-c)}{ad-e}, \delta\right).$$

We have

$$\dot{V} \leq D_9\,[|y| + |z| + |w| + |u|]\,[1 + |y| + |z| + |w| + |u|]\,q(t).$$

Using the inequalities

$|y| \leq 1+y^2$, $|z| \leq 1+z^2$, $|w| \leq 1+w^2$, $|u| \leq 1+u^2$, $2|yz| \leq y^2+z^2$, $2|yw| \leq y^2+w^2$, $2|yu| \leq y^2+u^2$, $2|zw| \leq z^2+w^2$, $2|zu| \leq z^2+u^2$, $2|wu| \leq w^2+u^2$

we get

$$\dot{V} \leq D_9\,[4 + 5(y^2+z^2+w^2+u^2)]\,q(t). \tag{2.14}$$

From the estimation given by (2.3) it is clear that

$$V \geq D_0\,(y^2+z^2+w^2+u^2), \tag{2.15}$$

where $D_0 = \min\,(D_2, D_3, D_4, D_5)$.

Thus it follows from (2.14) and (2.15) that

$$\dot{V} \leq D_{10}\,q(t) + D_{11}\,V q(t), \tag{2.16}$$

where $D_{10} = 4D_9$, $D_{11} = \dfrac{5D_9}{D_0}$.

Now, integrating (2.16) from 0 to t, we find

$$V(t) - V(0) \leq D_{10}\int_0^t q(s)ds + D_{11}\int_0^t V(s)\,q(s)ds.$$

Using condition (ii) of Theorem 2, we have

$$V(t) \leq D_{12} + D_{11}\int_0^t V(s)\,q(s)ds,$$

where $D_{12} = V(0) + D_{10}A$. Hence Gronwall-Bellman inequality yields

$$V(t) \leq D_{12}\,\exp\left(D_{11}\int_0^t q(s)ds\right).$$

This completes the proof of Theorem 2.

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