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ON THE STABILITY AND THE BOUNDEDNESS PROPERTIES OF SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

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Summary : The main purpose of this paper is to study the asymptotic stability in the large of the zero solution for Eq. (1.1) with $p \equiv 0$ and the boundedness of solutions for Eq. (1.1) with $p \neq 0$.

4. MERTEBEDEN BELİRLİ DİFERANSİYEL DENKLEMLERİN STABİLİTE VE SINIRLILIK ÖZELİKLERİ HAKKINDA

Özet : Bu çalışmanın ana amacı, $p \equiv 0$ halinde (1.1) denkleminin sıfır çözümünün asimtotik stabilitesini ve $p \neq 0$ halinde (1.1) in çözümlerinin sınırlılığını incelemektir.

1. Introduction and statement of the results

We consider the equation

$$x^{(4)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x})\dddot{x} + f(\dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \quad (1.1)$$

in which the functions φ, f, g, h and p depend at most on the arguments shown explicitly and the dots denote differentiation with respect to t. Further, it will be assumed that the functions φ, f, g, h and p are continuous for all values of their respective arguments and that the derivatives

$$\frac{\partial}{\partial x}\varphi(x, y, z, u), \frac{\partial}{\partial y}\varphi(x, y, z, u), \frac{\partial}{\partial u}\varphi(x, y, z, u), \frac{\partial}{\partial y}f(y, z), \frac{\partial}{\partial x}g(x, y),$$

$\frac{\partial}{\partial y}g(x, y)$ and $h'(x)$ exist and are continuous for all x, y, z and u. All functions and solutions are supposed to be real. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed.

Key words : Nonlinear differential equations of the fourth order, V-function, Stability, Boundedness.

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It will be convenient in what follows to use the equivalent system:

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,$$
$$\dot{u} = -\varphi(x, y, z, u)\,u - f(y, z) - g(x, y) - h(x) + p(t, x, y, z, u), \tag{1.2}$$

which is obtained from (1.1) by setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$.

The boundedness and stability properties of solutions for various equations of the fourth order differential equations have been considered by many authors. Many of these results are summarized in [12].

Ezeilo [4] investigated the stability and boundedness of the solutions of the equation

$$x^{(4)} + f(\ddot{x})\,\dddot{x} + \alpha_2\,\ddot{x} + g(\dot{x}) + \alpha_4\,x = p(t).$$

Harrow ([6], [7], [8]) studied the problem for the simple variant of (1.1) given by

$$x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t).$$

In [9], Lalli and Skrapek obtained a similar result for the equation

$$x^{(4)} + f_1(\ddot{x})\,\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x) = p(t).$$

Abou-El-Ela [1] investigated the boundedness of the solutions of the equation

$$x^{(4)} + f(\dot{x}, \ddot{x})\,\dddot{x} + \alpha_2\,\ddot{x} + g(\dot{x}) + \alpha_4\,x = p(t).$$

Also recently, in [3], Bereketoğlu dealt with the equation of the form

$$x^{(4)} + f_1(\dot{x}, \ddot{x}, \dddot{x})\,\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x) = p(t). \tag{1.3}$$

He presented sufficient conditions for the asymptotic stability in the large of the trivial solution of (1.3) with $p(t) \equiv 0$ and the boundedness of solutions of (1.3) with $p(t) \neq 0$.

In the case $p(t, x, y, z, u) \equiv 0$ we have

Theorem **1.** Suppose the following conditions are satisfied:

(i) $f(y, 0) = g(x, 0) = h(0) = 0$.

(ii) There are positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and Δ_0 such that

$$\alpha_1\alpha_2\alpha_3 - \alpha_3\frac{g(x, y)}{y} - \alpha_1\alpha_4\,\varphi(x, y, z, 0) \geq \Delta_0 \text{ for all } x, z \text{ and } y \neq 0.$$

(iii) $\varphi(x, y, z, u) \geq \alpha_1 > 0$ for all x, y, z and u,

$$\frac{f(y, z)}{z} \geq \alpha_2 \text{ for all } y, z \neq 0,$$

$$\frac{g(x,y)}{y} \geq \alpha_3 \text{ for all } x, y \neq 0,$$

$$\frac{h(x)}{x} \geq \beta \text{ for all } x \neq 0, \text{ where } \beta \text{ is a positive constant.}$$

(iv) $\left(\alpha_4 - \frac{\alpha_1 \Delta_0}{4\alpha_3}\right) < h'(x) \leq \alpha_4$ for all x.

(v) $\left(\frac{\partial}{\partial y} g(x,y) - \frac{g(x,y)}{y}\right) \leq \delta_1$ for all $x, y \neq 0$, where δ_1 is a positive constant satisfying $\delta_1 < \frac{2\alpha_4 \Delta_0}{\alpha_1 \alpha_3^2}$.

(vi) $\left(\frac{1}{z}\right)\int_0^z \varphi(x,y,s,0)\,ds - \varphi(x,y,z,0) \leq \delta_2$ for all x, y and $z \neq 0$, where δ_2 is a positive constant such that $\delta_2 < \frac{2\Delta_0}{\alpha_1^2 \alpha_3}$.

(vii) $\frac{\partial}{\partial y} f(y,z) \leq 0,\ y\frac{\partial}{\partial x}\varphi(x,y,z,0) \leq 0,\ z\frac{\partial}{\partial x}\varphi(x,y,z,0) \leq 0,$

$y\frac{\partial}{\partial y}\varphi(x,y,z,0) \leq 0$ and $z\frac{\partial}{\partial y}\varphi(x,y,z,0) \leq 0$ for all x, y and z.

(viii) $\frac{f(y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0 \alpha_3^3}{\alpha_4^2}$ for all $y, z \neq 0$, where ε_0 is a positive constant such that

$$\varepsilon_0 < \varepsilon \leq \min\left[\frac{1}{\alpha_1}, \frac{\alpha_4}{\alpha_3}, \frac{\Delta_0}{4\alpha_1 \alpha_3 D_0}, \frac{\alpha_3}{4\alpha_4 D_0}\left(\frac{2\alpha_4 \Delta_0}{\alpha_1 \alpha_3^2} - \delta_1\right), \frac{\alpha_1}{4D_0}\left(\frac{2\Delta_0}{\alpha_1^2 \alpha_3} - \delta_2\right)\right] \tag{1.4}$$

with $D_0 = \alpha_1 \alpha_2 + \frac{\alpha_2 \alpha_3}{\alpha_4}$.

(ix) $\left[\frac{\partial}{\partial x} g(x,y)\right]^2 \leq \frac{\alpha_1 \Delta_0 (\varepsilon - \varepsilon_0)}{16}$ for all x and y,

and $\frac{1}{y}\int_0^y \frac{\partial}{\partial x} g(x,s)\,ds \leq \frac{\sigma_3(\varepsilon - \varepsilon_0)}{4}$ for all $x, y \neq 0$.

(x) $z\frac{\partial}{\partial u}\varphi(x, y, z, u) + d_2 y \frac{\partial}{\partial u}\varphi(x, y, z, u) \geq 0$ for all x, y, z and u, where

$$d_2 = \frac{\alpha_4}{\alpha_3} + \varepsilon. \tag{1.5}$$

Then every solution of (1.1) satisfies

$$x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0, \dddot{x}(t) \to 0 \text{ as } t \to \infty. \tag{1.6}$$

In the case $p(t, x, y, z, u) \neq 0$ we have

Theorem 2. Suppose that the conditions of Theorem 1 hold and furthermore, the function $p(t, x, y, z, u)$ satisfies

$$|p(t, x, y, z, u)| \leq (1 + |y| + |z| + |u|)\, q(t), \text{ for all } t, x, y, z \text{ and } u, \tag{1.7}$$

where $q(t)$ is a nonnegative and continuous function of t, and satisfies

$$\int_0^t q(s)\, ds \leq A < \infty,$$

for all $t \geq 0$, with a positive constant. A. Then for any given finite x_0, y_0, z_0, u_0, there exists a constant $D \equiv D(x_0, y_0, z_0, u_0)$, such that the unique solution $x(t)$ of (1.1) is determined by the initial conditions

$$x(0) = x_0, \dot{x}(0) = y_0, \ddot{x}(0) = z_0, \dddot{x}(0) = u_0, \tag{1.8}$$

and it satisfies for all $t \geq 0$,

$$|x(t)| \leq D, |\dot{x}(t)| \leq D, |\ddot{x}(t)| \leq D, |\dddot{x}(t)| \leq D. \tag{1.9}$$

Remark 1. When $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}) = \alpha_1$, $f(\dot{x}, \ddot{x}) = \alpha_2 \ddot{x}$, $g(x, \dot{x}) = \alpha_3 \dot{x}$ and $h(x) = \alpha_4 x$, then equation (1.1) reduces to the linear differential equation with constant coefficients and conditions (i)-(x) of Theorem 1 reduce to the corresponding of Routh-Hurwitz criterion.

Remark 2. When we take $\varphi(x, \dot{x}, \ddot{x}, \dddot{x}) = f_1(\dot{x}, \ddot{x}, \dddot{x})$, $g(x, \dot{x}) = f_3(\dot{x})$ and finally $p(t, x, \dot{x}, \ddot{x}, \dddot{x}) = p(t)$, then conditions of Theorem 1 and Theorem 2 are reduced to theose of Bereketoğlu [3]. When $\varphi(x, \dot{x}, \ddot{x}, \dddot{x})$ and $g(x, \dot{x})$ depend only on $\ddot{x}$ and $\dot{x}$, respectively, and $f(\dot{x}, \ddot{x}) = \alpha_2 \ddot{x}$, $h(x) = \alpha_4 x$ and

$$p(t, x, \dot{x}, \ddot{x}, \dddot{x}) = p(t),$$

then conditions of Theorem 1 and Theorem 2 are reduced completely to those of Ezeilo [4]. Moreover, conditions of Theorem 1 and Theorem 2 reduce to the conditions of the relevant theorems by Lalli and Skrapek [9] and Harrow [6], up to very small differences. These differences are due to the fact that the Lyapunov function is not identical.

2. The Function $V(x, y, z, u)$

The main tool, in the proof of the theorems, is the function $V=V(x, y, z, u)$ defined by:

$$2V=2d_2\int_0^x h(s)\,ds+[d_2\alpha_2-d_1\alpha_4]\,y^2+2\int_0^y g(x,s)\,ds+2\int_0^z\Big[d_1 f(y,s)-d_2 s\Big]ds+$$
$$+2\int_0^z s\varphi(x,y,s,0)\,ds+2\,d_2 y\int_0^z \varphi(x,y,s,0)\,ds+d_1 u^2+2yh(x)+ \tag{2.1}$$
$$+2d_1 zh(x)+2d_1 z\,g(x,y)+2d_2 yu+2zu,$$

where

$$d_1=\frac{1}{\alpha_1}+\varepsilon, \tag{2.2}$$

d_2 being the constant defined by (1.5).

First discuss some important inequalities.

Let Φ_1 be the function defined by

$$\Phi_1(x,y,z,0)=\begin{cases}\left(\dfrac{1}{z}\right)\displaystyle\int_0^z \varphi(x,y,s,0)\,ds, & z\neq 0\\ \varphi(x,y,0,0), & z=0.\end{cases} \tag{2.3}$$

Using (iii) and (vi) we obtain

$$\Phi_1(x,y,z,0)\geq\alpha_1>0 \text{ for all } x, y \text{ and } z, \tag{2.4}$$

$$\Phi_1(x,y,z,0)-\varphi(x,y,z,0)\leq\delta_2 \text{ for all } x, y \text{ and } z. \tag{2.5}$$

Further we define

$$\Phi_3(x,y)=\begin{cases}\dfrac{g(x,y)}{y}, & y\neq 0\\ \dfrac{\partial}{\partial y}g(x,0), & y=0.\end{cases} \tag{2.6}$$

We have from (iii) and (v)

$$\Phi_3(x,y)\geq\alpha_3 \text{ for all } x \text{ and } y, \tag{2.7}$$

$$\frac{\partial}{\partial y}g(x,y)-\Phi_3(x,y)\leq\delta_1 \text{ for all } x \text{ and } y. \tag{2.8}$$

From (2.2) and (1.5) we have

$$\alpha_2 - d_1 \frac{g(x,y)}{y} - d_2\, \varphi(x,y,z,0) =$$

$$= \left(\frac{1}{\alpha_1 \alpha_3}\right)\left[\alpha_1 \alpha_2 \alpha_3 - \alpha_3 \frac{g(x,y)}{y} - \alpha_1 \alpha_4\, \varphi(x,y,z,0)\right] - \varepsilon \left[\frac{g(x,y)}{y} + \varphi(x,y,z,0)\right].$$

But also (ii) and (iii) imply that

$$\frac{g(x,y)}{y} < \alpha_1 \alpha_2 \,,\ \varphi(x,y,z,0) < \frac{\alpha_2 \alpha_3}{\alpha_4}.$$

Thus it follows that

$$\alpha_2 - d_1 \frac{g(x,y)}{y} - d_2\, \varphi(x,y,z,0) > \left(\frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0\right) \text{ for all } x, z \text{ and } y \neq 0, \tag{2.9}$$

by using (ii) and (viii).

Since $\Phi_1(x, y, z, 0) = \varphi(x, y, \tilde{z}, 0)$, $\tilde{z} = \theta z$, $0 \leq 0 \leq 1$, then

$$\alpha_3 - d_1 \frac{g(x,y)}{y} - d_2\, \Phi_1(x,y,z,0) \geq \frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0 . \tag{2.10}$$

The following two lemmas are to prove that the function $V(x, y, z, u)$ is a Lyapunov function of the system (1.2).

Lemma 1. Suppose that the conditions of Theorem 1 hold. Then there is a positive constant D_1 such that

$$V \geq D_1 [x^2 + y^2 + z^2 + u^2] \tag{2.11}$$

for all x, y, z and u.

Proof. $V(0, 0, 0, 0) = 0$, since $f(0, 0) = g(0, 0) = h(0) = 0$. Rewrite the function $2V(x, y, z, u)$ as follows:

$$\begin{aligned} 2V(x,y,z,u) = {} & \frac{1}{\Phi_1(x,y,z,0)} [u + z\,\Phi_1(x,y,z,0) + d_2\, y\, \Phi_1(x,y,z,0)]^2 + \\ & + \frac{1}{\Phi_3(x,y)} [h(x) + y\,\Phi_3(x,y) + d_1\, z \Phi_3(x,y)]^2 + V_1 + V_2 + V_3 + V_4 , \end{aligned} \tag{2.12}$$

where

$$V_1 = [d_2\, \alpha_2 - d_1\, \alpha_4 - d_2^2\, \Phi_1(x,y,z,0)]\, y^2 + 2\int_0^y g(x,s)\, ds - y^2\, \Phi_3(x,y),$$

$$V_2 = 2\,d_1 \int_0^z [f(y, s) - \alpha_2 s]\, ds + [d_1 \alpha_2 - d_2 - d_1^2 \Phi_3 (x, y)]\, z^2 +$$

$$+ 2 \int_0^z s\varphi (x, y, s, 0)\, ds - z^2 \Phi_1 (x, y, z, 0),$$

$$V_3 = 2d_2 \int_0^x h(s)\, ds - \frac{1}{\Phi_3(x, y)} \left[\frac{h(x)}{x} \right]^2 x^2 ,$$

$$V_4 = \left[d_1 - \frac{1}{\Phi_1 (x, y, z, 0)} \right] u^2 .$$

From (1.5), (2.2), (iii) and (2.10) we obtain

$$d_2 \alpha_2 - d_1 \alpha_4 - d_2^2 \Phi_1 (x, y, z, 0) > \frac{\alpha_4}{\alpha_3} \left(\frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) .$$

Since $yg (x, y) = \int_0^y g (x, \eta)\, d\eta + \int_0^y \eta g (x, \eta)\, d\eta$, then

$$2 \int_0^y g (x, \eta)\, d\eta - y^2 \Phi_3 (x, y) \geq \left(- \frac{\delta_1}{2} \right) y^2 , \text{ by (2.8).}$$

Therefore we get

$$V_1 \geq \frac{1}{2} \left[\frac{2\alpha_4 \Delta_0}{\alpha_1 \alpha_3^2} - \frac{2\alpha_4 D_0}{\alpha_3} \varepsilon - \delta_1 \right] y^2 > \frac{1}{4} \left[\frac{2\alpha_4 \Delta_0}{\alpha_1 \alpha_3^2} - \delta_1 \right] y^2 , \text{ by (1.4).}$$

By similar estimation, using condition (iii), (1.5), (2.2) and (2.9) we get

$$d_1 \alpha_2 - d_2 - d_1^2 \Phi_1 (x, y) =$$
$$= d_1 [\alpha_2 - d_1 \Phi_1 (x, y) - d_2 \varphi (x, y, z, 0)] + d_2 [d_1 \varphi (x, y, z, 0) - 1] >$$
$$> d_1 [\alpha_2 - d_1 \Phi_3 (x, y) - d_2 \varphi (x, y, z, 0)] > \left(\frac{1}{\alpha_1} \right) \left[\frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0 \right] . \quad (2.13)$$

From the identity

$$\int_0^z s\varphi (x, y, s, 0)\, ds = z \int_0^z \varphi (x, y, s, 0)\, ds - \int_0^z s\Phi_1 (x, y, s, 0)\, ds$$

we get

$$2\int_0^z s\varphi(x,y,s,0)\,ds - z^3\Phi_1(x,y,z,0) =$$

$$= \left[\int_0^z \{\varphi(x,y,s,0) - \Phi_1(x,y,s,0)\}\right] s\,ds \geq -\left(\frac{\delta_2}{2}\right) z^3, \text{ by (2.5).}$$

Also from (iii) we obtain

$$\int_0^z \left[\frac{f(y,s)}{s} - \alpha_2\right] s\,ds \geq 0.$$

Therefore

$$V_2 \geq \left\{\frac{1}{\alpha_1}\left(\frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon D_0\right) - \frac{\delta_2}{2}\right\} z^3 \geq \frac{1}{4}\left(\frac{2\Delta_0}{\alpha_1^2\alpha_3} - \delta_2\right) z^3, \text{ by (1.4).}$$

For the component V_3, from (i), (iii), (iv) and (1.5) it follows that

$$V_3 \geq 2(\varepsilon + \alpha_4\alpha_3^{-1}) \int_0^x h(s)\,ds - \frac{1}{\alpha_3}\left[\frac{h(x)}{x}\right]^2 x^2 \geq (\varepsilon\beta)\,x^2 +$$

$$+ 2\int_0^x \frac{h(s)}{s}\left[\frac{\alpha_4}{\alpha_3} - \frac{1}{\alpha_3} h'(s)\right] s\,ds \geq (\varepsilon\beta)\,x^2 .$$

By using (2.2) and (2.4) we obtain $V_4 \geq \varepsilon u^2$.

Combining the estimates for V_1, V_2, V_3 and V_4 with (2.12) we have

$$2V \geq (\varepsilon\beta)\,x^2 + \frac{1}{4}\left[\frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2} - \delta_1\right] y^2 + \frac{1}{4}\left(\frac{2\Delta_0}{\alpha_1^2\alpha_3} - \delta_2\right) z^2 + \varepsilon u^2,$$

noting that all the four coefficients of the above expression are nonnegative. Then there exists a positive constant D_1 such that

$$V \geq D_1\,[x^2 + y^2 + z^2 + u^2].$$

Thus the proof is now complete.

Lemma 2. Suppose that the conditions of Theorem 1 hold. Then there is a positive constant D_2 such that whenever (x, y, z, u) is any solution of (1.2) with $p(t, x, y, z, u) = 0$, then

$$\dot{V} \equiv \frac{d}{dt} V(x,y,z,u) \leq -D_2\,(y^2 + z^2 + u^2). \qquad (2.14)$$

Proof. A straightforward calculation using the identity

$$\frac{d}{dt} V = \frac{\partial V}{\partial u} \dot{u} + \frac{\partial V}{\partial z} u + \frac{\partial V}{\partial y} z + \frac{\partial V}{\partial x} y$$

yields

$$\dot{V} = - d_1 u^2 \varphi (x, y, z, u) - d_2 yf(y, z) - d_2 yg(x, y) - zf(y, z) + u^2 +$$

$$+ d_1 z \int_0^z \frac{\partial}{\partial y} f(y, s)\, ds + d_2 y^2 \int_0^z \frac{\partial}{\partial x} \varphi (x, y, s, 0)\, ds + z \int_0^z s \frac{\partial}{\partial y} \varphi (x, y, s, 0)\, ds +$$

$$+ d_2 yz \int_0^z \frac{\partial}{\partial y} \varphi (x, y, s, 0)\, ds + d_2 z \int_0^z \varphi (x, y, s, 0)\, ds\, [d_2 \alpha_2 - d_1 \alpha_4]\, yz +$$

$$+ d_1 yz \frac{\partial}{\partial x} g (x, y) + d_1 z^2 \frac{\partial}{\partial y} g (x, y) + y \int_0^y \frac{\partial}{\partial x} g (x, s)\, ds + y^2 h' (x) +$$

$$+ d_1 yzh' (x) - [\varphi (x, y, z, u) - \varphi (x, y, z, 0)]\, zu - d_2 [\varphi (x, y, z, u) - \varphi (x, y, z, 0)]\, yu +$$

$$+ y \int_0^z s \frac{\partial}{\partial x} \varphi (x, y, s, 0)\, ds.$$

Since

$$z \int_0^z \frac{\partial}{\partial y} f(y, s)\, ds \leq 0,\; y \int_0^z s \frac{\partial}{\partial x} \varphi (x, y, s, 0)\, ds \leq 0,\; z \int_0^z s \frac{\partial}{\partial y} \varphi (x, y, s, 0)\, ds \leq 0,$$

$$z \int_0^z y \frac{\partial}{\partial y} \varphi (x, y, s, 0)\, ds \leq 0 \text{ and } \int_0^z \frac{\partial}{\partial x} \varphi (x, y, s, 0)\, ds \leq 0, \text{ by (vii)},$$

then we obtain

$$\dot{V} \leq - \left[\alpha_2 - d_1 \frac{\partial}{\partial y} g (x, y) - d_2 \Phi_1 (x, y, z, 0) \right] z^2 - - [d_1 \varphi (x, y, z, u) - 1]\, u^2 - V_5 - V_6 - V_7 - V_8\,, \tag{2.15}$$

where

$$V_5 = f(y,z)\,z + d_2 f(y,z)\,y - \alpha_2 z^2 - \alpha_2 d_2 yz\,,$$

$$V_6 = [\varphi(x,y,z,u) - \varphi(x,y,z,0)]\,zu + d_2 [\varphi(x,y,z,u) - \varphi(x,y,z,0)]\,yu\,,$$

$$V_7 = \left[d_2 \frac{g(x,y)}{y} - \alpha_4\right] y^2 - d_1 yz \frac{\partial}{\partial x} g(x,y) - y \int_0^y \frac{\partial}{\partial x} g(x,s)\,ds\,, \qquad (2.16)$$

$$V_8 = (\alpha_4 - h'(x))\,y^2 + d_1 [\alpha_4 - h'(x)]\,yz\,.$$

By the same way as in (2.13), it follows that

$$\alpha_2 - d_1 \frac{\partial}{\partial y} g(x,y) - d_2 \Phi_1(x,y,z,0) \geq \left(\frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0\right) > \frac{3\Delta_0}{4\alpha_1 \alpha_3}\,,\ \text{by (1.4)}. \qquad (2.17)$$

By using (iii) and (2.2) we find

$$[d_1 \varphi(x,y,z,u) - 1] \geq \varepsilon \alpha_1\,. \qquad (2.18)$$

The function V_5 is the same as in [3]. The estimates for V_3 as in [3] give that

$$V_5 \geq -(\varepsilon_0 \alpha_3)\,y^2\,. \qquad (2.19)$$

Also, from (x) we obtain for $u \neq 0$

$$V_6 = [z\varphi_u(x,y,z,\theta u) + d_2 y \varphi_u(x,y,z,\theta u)]\,u^2 \geq 0,\ 0 \leq \theta \leq 1$$

but $V_6 = 0$ when $u = 0$. Hence

$$V_6 \geq 0 \text{ for all } x, y, z \text{ and } u. \qquad (2.20)$$

Combining (2.16) and (2.19) we obtain

$$V_5 + V_7 \geq -(\varepsilon_0 \alpha_3)\,y^2 + \left[d_2 \frac{g(x,y)}{y} - \alpha_4\right] y^2 - d_1 yz \frac{\partial}{\partial x} g(x,y) - y \int_0^y \frac{\partial}{\partial x} g(x,s)\,ds$$

$$\geq (\varepsilon - \varepsilon_0)\,\alpha_3 y^2 - d_1 yz \frac{\partial}{\partial x} g(x,y) - y \int_0^y \frac{\partial}{\partial x} g(x,s)\,ds$$

$$\geq (\varepsilon - \varepsilon_0)\,\alpha_1 y^2 - d_1 yz \frac{\partial}{\partial x} g(x,y) - \left[\frac{1}{y} \int_0^y \frac{\partial}{\partial x} g(x,s)\,ds\right] y^2$$

$$\geq \frac{3}{4} (\varepsilon - \varepsilon_0)\,\alpha_3 y^2 - d_1 yz \frac{\partial}{\partial x} g(x,y)$$

$$= \frac{1}{2} (\varepsilon - \varepsilon_0)\,\alpha_3 y^2 + \frac{1}{4} (\varepsilon - \varepsilon_0)\,\alpha_3 \left[y^2 - \frac{4d_1}{(\varepsilon - \varepsilon_0)\,\alpha_3} yz \frac{\partial}{\partial x} g(x,y)\right]$$

$$\geq \frac{1}{2}(\varepsilon - \varepsilon_0)\,\alpha_3\, y^2 - \frac{d_1^2}{(\varepsilon - \varepsilon_0)\,\alpha_3}\left[\frac{\partial}{\partial x} g(x, y)\right]^2 z^2$$

$$\geq \frac{1}{2}(\varepsilon - \varepsilon_0)\,\alpha_3\, y^2 - \frac{\Delta_0}{4\alpha_1\,\alpha_3} z^2, \tag{2.21}$$

by using (iii), (1.5), (ix), (2.2) and (1.4).

Now

$$V_8 = (\alpha_4 - h'(x))\,(y^2 + d_1\, yz) \geq -(\alpha_4 - h'(x))\,\frac{d_1^2}{4} z^2$$

$$> -\frac{\alpha_1\,\Delta_0}{16\,\alpha_3}\left(\frac{1}{\alpha_1} + \varepsilon\right)^2 z^2 > -\frac{\Delta_0}{4\alpha_1\,\alpha_3} z^2, \tag{2.22}$$

by using (iv), (2.2) and (1.4).

On gathering the estimates (2.17)-(2.22) into (2.15) we deduce that

$$\dot{V} \leq -\left(\frac{\Delta_0}{4\alpha_1\,\alpha_3}\right) z^2 - \frac{1}{2}(\varepsilon - \varepsilon_0)\,\alpha_3\, y^2 - (\varepsilon\alpha_1)\, u^2 \leq -D_2\,(y^2 + z^2 + u^2),$$

where $D_2 = \min\left\{\frac{\Delta_0}{4\alpha_1\,\alpha_3}, \frac{1}{2}(\varepsilon - \varepsilon_0)\,\alpha_3, \varepsilon\alpha_1\right\}$.

3. Proof of Theorem 1

By Lemma 1

$$V(x, y, z, u) = 0, \text{ at } x^2 + y^2 + z^2 + u^2 = 0,$$

$$V(x, y, z, u) > 0, \text{ if } x^2 + y^2 + z^2 + u^2 \neq 0$$

$$V(x, y, z, u) \to \infty, \text{ as } x^2 + y^2 + z^2 + u^2 \to \infty.$$

Also, let $(x(t), y(t), z(t), u(t))$ be any solution of (1.2) with $p(t, x, y, z, u) = 0$, such that $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $u(0) = u_0$. Consider the function $V(t) \equiv V(x(t), y(t), z(t), u(t))$ corresponding to this solution. By Lemma 2, we have

$$V(t) \leq V(0) \text{ for } t \geq 0.$$

Thus, the remainder of the proof of Theorem 1 is the same as the one given by Ezeilo [4] and hence is omitted.

4. Proof of Theorem 2

The proof here is based essentially on the method devised by Antosiewicz [2]. Let $(x(t), y(t), z(t), u(t))$ be the solution of (1.2) satisfying the initial conditions (1.8) and consider the function $V(t) \equiv V(x(t), y(t), z(t), u(t))$, where $V(x, y, z, u)$ is the function V used in the proof of Theorem 1. Using this function, we have that, for the system (1.2),

$$\dot{V} \leq -D_2(y^2 + z^2 + u^2) + (d_2 y + z + d_1 u)\, p(t, x, y, z, u),$$

so that

$$\dot{V} \leq D_3(|y| + |z| + |u|)\, |p(t, x, y, z, u)|,$$

where $D_3 = \max\{d_2, 1, d_1\}$.

It follows from (1.7) and the obvious inequalities

$$|y| \leq 1 + y^2, |z| \leq 1 + z^2, |u| \leq 1 + u^2, 2|yz| \leq y^2 + z^2,$$
$$|yu| \leq y^2 + u^2, |zu| \leq z^2 + u^2,$$

that

$$\dot{V} \leq D_3 [3 + 4(y^2 + z^2 + u^2)]\, q(t).$$

By (2.11) we have

$$V \geq D_1 [y^2 + z^2 + u^2].$$

Putting $D_4 = 3D_3$, $D_5 = \dfrac{4D_3}{D_1}$ we obtain

$$\dot{V} - D_5 q(t) V \leq D_4 q(t).$$

Therefore we obtain the result

$$V(t) \leq \frac{1}{x(t)}\left(V(0) + D_4 \int_0^t q(s)\, x(s)\, ds\right),$$

where $x(t) \equiv \exp\left(-D_5 \int_0^t q(s)\, ds\right)$. Since $x(t) \leq 1$ for $t \geq 0$,

$$V(t) \leq (V(0) + D_4 A)\, e^{D_5 A},$$

where $V(0) = V(x(0), y(0), z(0), u(0))$. The proof of Theorem 2 is complete.

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