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A CHARACTERIZATION FOR CONVEX FUNCTIONS OF COMPLEX ORDER b

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Summary : The main purpose of this note is tö give a new characterization for convex functions of complex order b. It should be noticed that by giving specific values to b we obtain new characterizations of important subclasses of convex functions of complex order b.

KOMPLEKS MERTEBEDEN KONVEKS FONKSİYONLAR İÇİN BİR KARAKTERİZASYON

Özet : Bu makalenin asıl amacı, kompleks mertebeden konveks fonksiyonlar için bir karakterizasyon vermektir. Fakat burada önemle belirtelim ki, bu temel karakterizasyondan $b \neq 0$ kompleks sayısı ve $-1<k<1$ olmak üzere k reel sayısının aldığı özel değerler için konveks, β'ıncı mertebeden konveks, $(zf'(z))$ λ-spirallike olan ve $(zf'(z))$ β'ıncı mertebeden λ-spirallike olan fonksiyon sınıfları için temel karakterizaşyonlar ve eşitsizlikler elde edilir. Bunların çoğu A.W. Goodman tarafından yazılmış "univalent functions" adlı kitapta açık problemler olarak belirtilmektedir. Ayrıca $b = 1, k = 0$ için 1948 yılında O. Hamdi Alisbah tarafından ortaya atılan "düzgün yalınkatlık" tanımında belirtilen $m>0$ sayısının konveks fonksiyonlar için $\frac{1}{2} = m$ olduğu da ispatlanmıştır.

1. INTRODUCTION

Let R denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ which are analytic in $D = \{z \mid |z| < 1\}$. A function $f(z)$ in R is said to be a convex function of complex order b ($b \neq 0$, complex), that is $f(z) \in C(b)$, if and only if $f'(z) \neq 0$ in D and

$$Re\left(1 + \frac{1}{b} z \frac{f''(z)}{f'(z)}\right) > 0, \quad z \in D. \tag{1.1}$$

The class $C(b)$ was introduced by P.Wiatrowski [5]. If we give specific values to b we obtain the following important subclasses:

(i) $C(1) = c$ is the well-known class of convex functions

(ii) $C(1-\beta), 0 < \beta < 1$, is the class of convex functions of order β

(iii) $C(e^{-i\lambda} \operatorname{Cos} \lambda), |\lambda| < \frac{\pi}{2}$, is the class of functions for which $zf'(z)$ is λ-spirallike

(iv) $C((1-\beta)\, e^{-i\lambda} \operatorname{Cos} \lambda),\ 0 \leq \beta < 1, |\lambda| < \frac{\pi}{2}$, is the class of functions for which $zf'(z)$ is λ-spirallike of order β.

Definition 1.1. Let $S(1-b)$ ($b \neq 0$, complex) be denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ in D, which satisfy for $z = r e^{i\theta} \in D$

$$Re\left[1 + \frac{1}{b}\left(z\,\frac{f'(z)}{f(z)} - 1\right)\right] > 0,\ z \in D, \tag{1.2}$$

then $f(z)$ are said to be starlike functions of complex order b. We note that if $b=1$, $S(1-1) = S(0)$ is well-known class of starlike functions.

Theorem 1.1. Let $f(z)$ be anaytic in D and normalized so that $f(0) = f'(0) - 1 = 0$. A necessary and sufficient condition for $f(z) \in S(0)$ is that for each real number k, $-1 < k < 1$, the function $F_k(z)$, defined by the equations

$$F_K(z) = \left[\frac{kf(z)}{f(kz)}\right]^{1/2},\ F_K(0) = 1,\ F_0(z) = \left[\frac{f(z)}{z}\right]^{1/2},$$

be analytic and subordinate to $\frac{1+kz}{1+z}$, $z \in D$, or equivalently that

$$Re\ F_K(z) > \frac{1+k}{2},\quad \left|\frac{1+k}{F_K(z)} - 1\right| < 1, z \in D, -1 < k < 1.$$

This theorem was proved by M.S.Robertson [3].

Theorem 1.2. Let $f(z)$ be regular in the unit circle and normalized so that $f(0) = f'(0) - 1 = 0$. A necessary and sufficient condition for $f(z) \in C(b)$ is that for each member $s(z) = z + b_2 z^2 + \in \ldots S(1-b)$ the equation

$$S(z) = z\left(\frac{f(z) - f(\eta)}{z - \eta}\right)^2,\quad z, \eta \in D, z \neq \eta$$

must be satisfied.

This theorem was proved by Y.Polatoğlu [6].

2. A CHARACTERIZATION FOR CONVEX FUNCTIONS OF COMPLEX ORDER b

Lemma 2.1. Let $f(z) = z + a_2 z^2 + \dots$ and $g(z) = z + b_2 z^2 + \dots$ be analytic in D. A necessary and sufficient condition for $g(z) \in S(1 - b)$ is that for each member $f(z) \in S(0)$ the equation

$$f(z) = z \left(\frac{g(z)}{z}\right)^{1/b}, \; z \in D$$

must be satisfied.

Proof. If we take the logarithmic derivative of (2.1) we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{1}{b}\,\frac{g'(z)}{g(z)} - \frac{1}{b}\,\frac{1}{z}. \tag{2.2}$$

From this we get

$$Re\; z\,\frac{f'(z)}{f(z)} = Re\left[1 + \frac{1}{b}\left(z\,\frac{g'(z)}{g(z)} - 1\right)\right] > 0\,. \tag{2.3}$$

Considering the relation (2.3) and the definition of starlike functions of complex order, we observe that this lemma is true.

Theorem 2.1. Let $s(z) = z + c_2 z^2 + \dots$ be analytic in D. A necessary and sufficient condition for $S(z) \in C(b)$ is that for each real number $k, -1 < k < 1$, the function $F(k, b, z, \eta)$, defined by the equations

$$F(k, b, z, \eta) = \left[\frac{k(s(z) - s(\eta))}{s(kz) - s(k\eta)}\right]^{1/b}, \tag{2.4}$$

$$F(k, b, 0, 0) = 1, \tag{2.5}$$

$$F(k, b, z, \eta) = \left[\frac{s(z) - s(\eta)}{z - \eta}\right]^{1/b} \tag{2.6}$$

be analytic and subordinate to $\dfrac{1 + kz}{1 + z}, z \in D$ or equivalently that

$$Re\,(k, b, z, \eta) > \frac{1 + k}{2}, \quad \left|\frac{1 + k}{F\,(k, b, z, \eta)} - 1\right| < 1. \tag{2.7}$$

Proof. Let $f(z) = z + a_2 z^2 + \dots$, $g(z) = z + b_2 z^2 + \dots$ and $S(z) = z + C_2 z^2 + \dots$ be regular in D and let $f(z) \in S(0)$, $g(z) \in S(1 - b)$ and $S(z) \in C(b)$.

Now considering Theorem 1.1, Theorem 1.2 and Lemma 2.1 all together, we conclude that

$$\left[\frac{kf(z)}{f(kz)}\right]^{1/2} = \left[\frac{kz\left(\frac{g(z)}{z}\right)^{1/b}}{kz\left(\frac{g(kz)}{kz}\right)^{1/b}}\right]^{1/2} = \left[\frac{kz\left(\frac{s(z)-s(\eta)}{z-\eta}\right)^{2}}{kz\left(\frac{s(kz)-s(k\eta)}{kz-k\eta}\right)^{2}}\right]^{1/2b}$$

$$= \left[\frac{k(s(z)-s(\eta))}{s(kz)-s(k\eta)}\right]^{1/b} = F(k, b, z, \eta), \tag{2.8}$$

$$F(b, k, 0, 0) = 1,\ F(0, b, z, \eta) = \left[\frac{s(z)-s(\eta)}{z-\eta}\right]^{1/b},$$

where $z, \eta \in D$, $z \neq \eta$, $b \neq 0$, b is complex and k is real and $-1 < k < 1$. Hence the proof of this theorem is complete.

Corollary 2.1. For $k = 0, b = 1$ we obtain

$$Re\, F(0, 1, z, \eta) = Re\left(\frac{s(z)-s(\eta)}{z-\eta}\right) > \frac{1}{2}, \tag{2.9}$$

$$0 < m = \frac{1}{2} < Re\left(\frac{s(z)-s(\eta)}{z-\eta}\right) < \left|\frac{s(z)-s(\eta)}{z-\eta}\right|. \tag{2.10}$$

The inequality (2.10) is answer to the O. Hamdi Alisbah's question for convex functions [4], and this inequality shows that all convex functions are regularly univalent functions in the unit disc with respect to the definition of regularly univalent functions. Regularly univalent functions are defined by O. Hamdi Alisbah [4].

Corollary 2.2. For $b = 1$, $k \to -1$ we get

$$Re\left[\frac{-s(z)+s(\eta)}{s(-z)-s(-\eta)}\right] = Re\, F(-1, 1, z, \eta) > 0.$$

This result is a new inequality for convex functions.

Corollary 2.3. By giving special values to b we obtain new characterizations and new inequalities for the important subclasses of convex functions of complex order b. These inequalities are open problems (see [1]).

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