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ON THE SERIES METHODS CONTAINING THE METHOD V_σ

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Abstract. *In [7], Raimi has defined the regular sequence methods containing the method V_σ and determined the necessary and sufficient conditions for a regular sequence method to contain the method V_σ. In this study we introduce the series methods containing the method V_σ and obtain the necessary and sufficient conditions for a regular series method to contain the method V_σ, and also prove that one of the dual summahility methods contains the method V_σ if and only if the other one contains the method V_σ.*

1. Introduction

Let m, c and c_0 be the Banach spaces of real or complex bounded, convergent and null sequences $x = (x_n)$ with the usual supremum norm. Let σ be a one-to-one mapping of the set of positive integers into itself. A continuous linear functional ϕ on m is said to be an invariant mean or a σ-mean if and only if (i) $\phi(x) \geq 0$ when $x_n \geq 0$ for all n, (ii) $\phi(e) = 1$ where $e = (1,1,1,...)$ and (iii) $\phi(Tx) = \phi(x)$ for all $x \in m$, where $T: m \to m$ is a linear operator defined by $Tx = x_{\sigma(n)}$.

Throughout this paper we consider that $\sigma^q(n) \neq n$ for all positive integers n and q, where $\sigma^q(n)$ is qth iterate of σ at n. Thus, a σ-mean extends the limit functional on c, in the sense that $\phi(x) = \lim x$ for all $x \in c$ [5]. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal.

In case $\sigma(n) = n+1$, the σ-means are the classical Banach limits on m and V_σ is the set of almost convergent sequences introduced by Lorentz [4].

It can be shown [7] that

$$V_\sigma = \left\{ x \in m : \lim_q t_{qn}(x) = L \text{ uniformly in n, } L = \sigma - \lim x \right\} \qquad ...(1)$$

where, for $q \geq 0$, $n > 0$,

$$t_{qn}(x) = (x + Tx + ... + T^q x)/(q+1).$$

The special case of (1) in which $\sigma(n) = n+1$ was given by Lorentz [4].

If w is a subset of the space of real sequences, Then we write w^+ for the generalized Köthe-Toeplitz dual of w, i.e.,

$$w^+ = \left\{ (a_k) : \sum_k a_k x_k \text{ converges for every } x \in w \right\}$$

It is well-known that $bs^+ = bv \cap c_0$ [2, p. 68]; where bs and bv are respectively the spaces of bounded sums and sequences of bounded variation.

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} (n, k=0,1,2...) and $x = (x_k)$ be a real sequence such that the series

$$A_n x = \sum_k a_{nk} x_k \qquad ...(2)$$

exists for each n. Then the sequence $Ax = (A_n(x))$ is the called A-transform of x. Hence, the summability method A is a sequence method. Throughout the paper $\sum_k$ will denote summation from k=0 to ∞. A sequence x is said to be A-summable if $Ax \in c$.

A matrix A is called regular if $Ax \in c$ and $\lim Ax = \lim x$ for all $x \in c$. The regularity conditions of A are well-known [1, p. 64]. A regular matrix A is called σ-invariant if $\lim A(T-I)x = 0$ for all $x \in m$; where I is the identity matrix. Raimi [7, Th. 23] proved that if a regular matrix A is σ-invariant, then

$$\lim_n \sum_k |a_{nk} - a_{n,\sigma(k)}| = 0. \qquad ...(3)$$

Conversely, if A is non negative, i.e. $a_{nk} \geq 0$ for all n, k, then the condition (3) is sufficient for σ-invariance of A.

2. The Dual Summability Methods

Let $A = (a_{nk})$ be a sequence method given by (2). Suppose that, for each n, the series

$$\sum_k a_{nk} \qquad ...(4)$$

is convergent; this is a much weaker assumption than the regularity of A. Then we shall define the matrix $B = (b_{nk})$ as follows:

$$b_{nk} = \sum_{i=k}^{\infty} a_{ni} \quad (n, k = 0,1,2...).$$

We also suppose that the sequence $s = (s_k)$ is formed by taking the partial sums of the series $\sum x_k$.

Let B denotes the summability method given by the series -to- sequence transformation (series method)

$$B_n(x) = \sum_k b_{nk} x_k \quad (n = 0,1,2...)$$

The methods A and B are called dual summability methods [3]. The method B is called regular if the sequence $Bx = (B_n(x)) \in c$ and $\lim Bx = \sum x_k$ for all $x \in cs$, the space of all convergent series. The regularity conditions of B are well-known[1, p. 68]. Moreover, it is also known that A is regular if and only if B is regular.

In the sequel, by A and B we mean the sequence and the series methods, respectively.

3. The Series Methods Containing The Method V_σ

We call the regular method A containing the method V_σ if the A-transform of x is convergent to the $\sigma-\lim x$ for each $x \in V_\sigma$, i.e., $Ax \in c$ and $\lim Ax = \sigma - \lim x$ for each $x \in V_\sigma$. Raimi has characterized the class of the regular sequence methods containing the method V_σ as follows:

Theorem **3.1** ([7], Th. **24**). A regular method A contains the method V_σ if and only if it is σ - invariant.

Similarly, we shall define the series methods containing the method V_σ and give the necessary and sufficient conditions for a regular method B to contain the method V_σ.

Definition 3.1. Let $\sum x_k$ be an infinite series such that its sequence of partial sums $s = (s_k) \in V_\sigma$ with $\sigma - \lim s = b$. If the method B sums every series $\sum x_k$ of this type, to the same value b then B is said to be contain the method V_σ.

Let

$$V_\sigma s = \{ x = (x_k) : (s_n) \in V_\sigma,\ s_n = \sum_{k=0}^{n} x_k \}.$$

If $\lim Bx = \sigma$ - lims for all $x \in V_\sigma s$, then we write $B \in (V_\sigma s, c)_{reg}$. Therefore B contains the method V_σ if and only if $B \in (V_\sigma s, c)_{reg}$.

Theorem 3.2. A regular method B contains the method V_σ if and only if

(i) $\lim_k b_{nk} = 0$ for each n,

(ii) $\lim_n \sum_k |\Delta(b_{nk} - b_{n,\sigma(k)})| = 0$

where $\Delta(b_{nk} - b_{n,\sigma(k)}) = b_{nk} - b_{n,\sigma(k)} - b_{n,k+1} + b_{n,\sigma(k+1)}$.

P r o o f: Necessity- Suppose that the regular method B contains the method V_σ. Since $V_\sigma s \subset bs$ and so $bs^+ \subset (V_\sigma s)^+$, the condition (i) must be satisfied, or else the series $\sum_k b_{nk} x_k$ diverges for at least one n which means that B-transform of $x \in V_\sigma s$ does not exist.

Now, by Abel's partial summation, we get that

$$\sum_{k=0}^{p} b_{nk} x_k = \sum_{k=0}^{p-1} (b_{nk} - b_{n,k+1}) s_k + b_{np} s_p \qquad ...(5)$$

for each n, where $x \in V_\sigma s$ and (s_p) is the sequence of pth partial sums of the series $\sum x_k$.

Since $(s_p) \in m$, we have from the condition (i) that $\lim_p b_{np} s_p = 0$. Hence, by letting $p \to \infty$ in (5), we obtain

$$\sum_{k=0}^{\infty} b_{nk} x_k = \sum_{k=0}^{\infty} (b_{nk} - b_{n,k+1}) s_k = \sum_{k=0}^{\infty} \Delta b_{nk} s_k \qquad ...(6)$$

which implies from the assumption that the method $A = (a_{nk}) = (Ab_{nk})$ contains the method V_σ. Therefore, the condition (ii) follows from Theorem 3.1 and (3).

Sufficiency- Suppose that the regular method B satisfies the conditions (i) and (ii). Let $x \in V_\sigma s$ with $\sigma - \lim s = b$. Then the regularity of B and (i) imply together that (6) holds. Since the method A defined as above is regular, it follows from the conditions (ii) that A is σ- invariant and so, by Theorem 3.1, A contains the method V_σ. Now, Letting $n \to \infty$ in (6), we get $\lim Bx = b$ which completes the proof.

We shall note that A must be non negative for the sufficiency part of the proof.

Theorem 3.3. Let A and B be two summability methods. Then A contains the method V_σ if and only if B contains the method V_σ

P r o o f: As it is well-known that A is regular if and only if B is regular. Moreover, since the series (4) is convergent for each n, (i) holds. On the other hand, A satisfies (3) if and only if B satisfies (ii).

Thus, the proof is completed.

Finally, we should declare that Theorem 3.2 and Theorem 3.3 are reduced to Theorem 2.1 and Theorem 2.2 of Öztürk [6] if $\sigma(n) = n + 1$.

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