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THE SPACE $W_w^p(IR)$ AND SOME PROPERTIES

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ABSTRACT

Assume that w is Beurling's weight on the real numbers IR and $1 \le p, q < \infty$. In this paper, we defined a weighted space

$$W_w^p(IR) = \left\{ f \in W^p(IR) \;\middle|\; \hat{f} \in L_w^q(IR) \right\}$$

and endowed it with the sum norm

$$\|f\|_w^p = \|f\|_{W^p} + \left\|\hat{f}\right\|_{q,w} .$$

We showed that $W_w^p(IR)$ is a Segal algebra. We also discussed the inclusions between the spaces $W_w^p(IR)$ and the multipliers from $L^1(IR)$ to $W_w^p(IR)$.

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1.INTRODUCTION

Throughout this work, G denotes a locally compact Abelian (non-discrete, non-compact) group with dual group $\hat{G}$ and dx, $d\hat{x}$ denotes Haar measure on G and $\hat{G}$, respectively. We will denote the space of all continuous, complex-valued function on G with compact support by $C_c(G)$ and the bounded regular Borel measures on G by $M(G)$. The translation operators L_x is given by $L_x f(y) = f(y-x)$ and the multiplication operator M_t is defined as $M_t f(y) = \langle t, y \rangle f(y)$ for $x, y \in G$ and $t \in \hat{G}$. $(B, \|.\|_B)$ is called strongly translation invariant if one has $L_x B \subseteq B$ and $\|L_x f\|_B = \|f\|_B$ for all $f \in B$ and

$x \in G$. Let $\left(B, \|.\|_B\right)$ be a Banach space. If B is strongly translation invariant and the map $x \to L_x f$ from G onto B is continuous, then we say that B is a Homogenous Banach space on G. Let B be subalgebra of $L^1(G)$. If B is a Homogenous Banach space, B is a Banach algebra with the norm $\|.\|_1 \leq \|.\|_B$ and also B is everywhere dense in $L^1(G)$ with respect to the norm $\|.\|_1$, then B is a called to be a Segal algebra. The spaces $L^1_{loc}(G)$ consists of all measurable functions f on G such that $f_{\chi_K} \in L^1(G)$ for any compact subset $K \subset G$, where χ_K is the characteristic function of K. A Banach function space(shortly BF-space) on G is a Banach space $\left(B, \|.\|_B\right)$ of measurable functions embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$, there exists some constant $c_K > 0$ with

$$\|f.\chi_K\|_1 \leq c_K \|f\|_B$$

for all $f \in B$. A Banach space $\left(B, \|.\|_B\right)$ is called a Banach module over a Banach algebra $\left(A, \|.\|_A\right)$, if B is a module over A in the algebraic sense for some multiplication, and the inequality $\|a.b\|_B \leq \|a\|_A \|b\|_B$ is satisfied. For a Beurling's weight w on G [6], i.e. a continuous function w satisfying $w(x) \geq 1$ and $w(x+y) \leq w(x).w(y)$ for all $x, y \in G$, we set

$$L^q_w(G) = \left\{ f \;\middle|\; f w \in L^q(G) \right\}$$

for $1 \leq q < \infty$. It is a Banach space under the norm

$$\|f\|_{q,w} = \left\{ \int_G |f(x) w(x)|^q \, dx \right\}^{1/q}$$

The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$f(y) = \int_G f(x) \langle -x, y \rangle \, dx \quad , y \in \hat{G}.$$

It is known that $\hat{f}$ is continuous and vanishes at infinity and also the inequality $\|\hat{f}\|_\infty \leq \|f\|_1$ is satisfied[6]. Lastly a space $W^p(IR^n)$ is defined to be

$$W^p(IR^n) := \left\{ f \in L^1(IR^n) \;\middle|\; \sum_{m \in Z^n} \|\chi_m f\|_p < \infty \right\}.$$

It is known that $W^p(IR)$ is a Banach algebra under convolution with the norm

$$\|f\|_{W^p} := \max_{t \in Q} \sum_{m \in Z} \|\chi_m f_t\|_p ,$$

where Q is the cube $\left\{x \in IR^n \;\middle|\; x = (x_i)_{i=1}^n, -\tfrac{1}{2} \leq x_i \leq \tfrac{1}{2}\right\}$ and $1 < p \leq \infty$ [4]. If B_1 and B_2 are Banach A-modules, then a multiplier(or module homomorphism) from B_1 to B_2 is a bounded linear operator T from B_1 to B_2 which commutes with module multiplication i.e. $T(a.b) = aT(b)$ for all $a \in A, b \in B_1$. We denote by $Hom_A(B_1, B_2)$ or $M(B_1, B_2)$ the space of multipliers from B_1 to B_2.

2.THE SPACES $W_w^p(IR)$

Let w be weight function on IR and $1 \leq p, q < \infty$. We define $W_w^p(IR)$ by

$$W_w^p(IR) = \left\{ f \in W^p(IR) \;\middle|\; \hat{f} \in L_w^q(IR) \right\}.$$

Since the spaces $W^p(IR)$ and $L_w^q(IR)$ are vector spaces, then it is easily to see that $W_w^p(IR)$ is also a vector space. It is also easily proved that $W_w^p(IR)$ is a subspace of $W^p(IR)$.

THEOREM 2.1 $W_w^p(IR)$ is a Banach algebra under convolution with the norm $\|f\|_w^p = \|f\|_{W^p} + \|\hat{f}\|_{q,w}$, $f \in W_w^p(IR)$.

PROOF: Suppose that $(f_n)_{n \in N}$ is a Cauchy sequence in $W_w^p(IR)$. Clearly $(f_n)_{n \in N}$ and $(\hat{f}_n)_{n \in N}$ are Cauchy sequences in $W^p(IR)$ and $L_w^q(IR)$, respectively. Since $W^p(IR)$ and

$\widehat{L_x f}(t) := \overline{\langle x,t\rangle}\hat{f}(t)$, one can obtain $\left\|(L_x f)^{\wedge}\right\|_{q,w} = \left\|\hat{f}\right\|_{q,w}$ for every $x,t \in IR$. Therefore $L_x f \in W_w^p(IR)$ and

$$\|L_x f\|_w^p = \|L_x f\|_{W^p} + \left\|(L_x f)^{\wedge}\right\|_{q,w} = \|f\|_{W^p} + \left\|\hat{f}\right\|_{q,w} = \|f\|_w^p.$$

PROPOSITION 2.3 The function $x \to L_x f$ is continuous from IR into $W_w^p(IR)$ for every $f \in W_w^p(IR)$.

PROOF: We known that the function $x \to L_x f$ is continuous from IR into $W^p(IR)$ [4], and the function $x \to \widehat{L_x f}$ is continuous from IR into $L_w^q(IR)$ in [3] for all $f \in W_w^p(IR)$. Let $f \in W_w^p(IR)$ and $\varepsilon > 0$ be given. Take any $x_0 \in IR$. Hence there are neighbourdhoos U_1 and U_2 of x_0 such that $\|L_x f - L_{x_0} f\|_{W^p} < \varepsilon/2$ and $\left\|\widehat{L_x f} - \widehat{L_{x_0} f}\right\|_{q,w} < \varepsilon/2$ for all $x \in U_1$ and $x \in U_2$. Then $\|L_x f - L_{x_0} f\|_w^p < \varepsilon$ for every $x \in U$.

PROPOSITION 2.4 $W_w^p(IR)$ is character invariant.

PROOF: Let $f \in W_w^p(IR)$ and $t_0 \in IR$ be given. We have

$$\|M_{t_0} f\|_{W^p} = \max_{t \in Q} \sum_{n \in Z} \left\|\chi_{Q_n}\left(M_{t_0}\right)_t\right\|_p = \max_{t \in Q} \sum_{n \in Z} \left\{\int_{Q_n} \left|M_{t_0} f(x-t)\right|^p dx\right\}^{1/p} =$$

$$= \max_{t \in Q} \sum_{n \in Z} \left\{\int_{Q_n} \left|(x-t,t_0) f(x-t)\right|^p dx\right\}^{1/p} =$$

$$= \max_{t \in Q} \sum_{n \in Z} \left\{\int_{Q_n} \left|f_t(x)\right|^p dx\right\}^{1/p} = \max_{t \in Q} \sum_{n \in Z} \left\|\chi_{Q_n} f_t\right\|_p = \|f\|_{W^p}. \tag{2.4.1}$$

Then $M_{t_0} \in W^p(IR)$. Also because the equality $\widehat{M_{t_0} f} = L_{t_0} \hat{f}$, one obtains

$L^q_w(IR)$ are Banaeh spaces, then there exist $f \in W^p(IR)$ and $g \in L^q_w(IR)$ such that $\|f_n - f\|_{W^p} \to 0$, $\|\hat{f}_n - g\|_{q,w} \to 0$. Using the inequalities $\|.\|_1 \le \|.\|_{W^p}$ and $\|.\|_q \le \|.\|_{q,w}$ one obtains $\|f_n - f\|_1 \to 0$, $\|\hat{f}_n - g\|_q \to 0$. Then there exists a subsequence $(\hat{f}_{n_k})$ of $(\hat{f}_n)_{n\in N}$ which converges to g almost everywhere(a.e.). It follows from the inequality $\|\hat{f}_n - \hat{f}\|_\infty \le \|f_n - f\|_{W^p}$ that $\|\hat{f}_n - \hat{f}\|_\infty \to 0$. Hence it is easily showed that $\|\hat{f}_{n_k} - \hat{f}\|_\infty \to 0$. Therefore $\hat{f} = g$. Thus $\|f_n - f\|^p_w \to 0$ and $f \in W^p_w(IR)$. That means $W^p_w(IR)$ is a Banach space. Now, let $f, g \in W^p_w(IR)$ be given. Since $W^p(IR)$ is a Banach algebra under convolution[4], then $f * g \in W^p(IR)$. If one uses the inequality

$$\left\| f \hat{*} g \right\|_{q,w} = \left\{ \int_{IR} \left| (f*g)^{\wedge}(x) w(x) \right|^q dx \right\}^{1/q} \le \|\hat{f}\|_\infty \|\hat{g}\|_{q,w}$$

obtains $f \hat{*} g \in L^q_w(IR)$. Thus $f * g \in W^p_w(IR)$. Also using the inequalities

$$\|f * g\|_{W^p} \le \|f\|_{W^p} \|g\|_{W^p},$$

$$\|\hat{f}\|_\infty \le \|f\|_1 \le \|f\|_{W^p},$$

we obtain

$$\|f * g\|^p_w = \|f * g\|_{W^p} + \left\| f \hat{*} g \right\|_{q,w} \le \|f\|_{W^p} \|g\|_{W^p} + \|f\|_1 \|\hat{g}\|_{q,w} \le \|f\|^p_w \|g\|^q_w$$

for all $f, g \in W^p_w(IR)$. Therefore $W^p_w(IR)$ is a Banach algebra.

PROPOSITION 2.2 The space $W^p_w(IR)$ is strongly translation invariant.

PROOF: It is known that $W^p(IR)$ is strongly translation invariant[4]. Hence $L_x f \in W^p(IR)$ and $\|L_x f\|_{W^p} = \|f\|_{W^p}$ for all $f \in W^p(IR)$. On the other hand, using

$$\left\| \widehat{M_{t_0} f} \right\|_{q,w} = \left\| L_{t_0} \hat{f} \right\|_{q,w} \le w(t_0) \left\| \hat{f} \right\|_{q,w}. \quad (2.4.2)$$

Therefore $\widehat{M_{t_0} f} \in L^q_w(IR)$. If one uses (2.4.1), (2.4.2) and the definition of the norm $\|\cdot\|^p_w$ finds that $M_{t_0} f \in W^p_w(IR)$.

PROPOSITION 2.5 $\left(W^p_w(IR), \|\cdot\|^p_w\right)$ is a Segal algebra.

PROOF: It is easy to see that $W^p_w(IR)$ is a subalgebra of $L^1(IR)$. Also $W^p(IR)$ is everywhere dense in $L^1(IR)$[4]. Then given any $f \in L^1(IR)$, there is $\varepsilon > 0$ a $g \in W^p(IR)$ such that

$$\|f - g\|_1 < \varepsilon/2. \quad (2.5.1)$$

Also since $W^p(IR)$ is a Segal algebra, then there exists $v \in W^p(IR)$ such that the Fourier transformation $\hat{v}$ of v has compact support and $\|v * g - g\|_{W^p} < \varepsilon/2$ [6]. Hence using the inequality $\|\cdot\|_1 \le \|\cdot\|_{W^p}$, we obtain

$$\|v * g - g\|_1 < \varepsilon/2. \quad (2.5.2)$$

Thus we have $v * g \in W^p(IR)$. We set $supp\hat{v} = K$. Because the inequality

$$\left\| \widehat{v * g} \right\|_{q,w} = \left\{ \int_{IR} \left| (v * g)^{\wedge}(x) w(x) \right|^q dx \right\}^{1/q} = \left\{ \int_K \left| \hat{v}(x) \hat{g}(x) w(x) \right|^q dx \right\}^{1/q} \le$$

$$\le \sup_{x \in K} \left| \hat{v}(x) \hat{g}(x) \right| \sup_{x \in K} |w(x)| \left\{ \int_K d\mu(x) \right\}^{1/q} = \|\hat{v}\,\hat{g}\|_\infty \|w\|_{\infty,K}\, \mu(K)$$

we have $\widehat{v * g} \in L^q_w(IR)$. Thus $v * g \in W^p_w(IR)$. Also using (2.5.1) and (2.5.2), we conclude that

$$\|f - v * g\|_1 \le \|f - g\|_1 + \|g - v * g\|_1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

i.e. the space $W_w^p(IR)$ is everywhere dense in $L^1(IR)$. Consequently, if one uses Proposition 2.1, Proposition 2.2 and Proposition 2.3 obtain that $W_w^p(IR)$ is a Segal algebra.

PROPOSITION 2.6 The space $W_w^p(IR)$ is a BF-space on IR.

PROOF: Let any compact subset K of IR and any $f \in W_w^p(IR)$ be given. Using the inequality $\|\cdot\|_1 \le \|\cdot\|_{W^p}$ and the definition of norm, we write

$$\int_K |f(x)|dx \le \int_{IR} |f(x)|dx \le \|f\|_w^p.$$

This completes the proof.

LEMMA 2.7 Let w_1, w_2 be two weights on IR and $1 \le p, q < \infty$. Then if $W_{w_1}^p(IR) \subset W_{w_2}^p(IR)$, then there exists a constant $a > 0$ such that $\|f\|_{w_2}^p \le a\|f\|_{w_1}^p$ for every $f \in W_{w_1}^p(IR)$.

PROOF: Suppose $W_{w_1}^p(IR) \subset W_{w_2}^p(IR)$. We define the norm $|||f||| = \|f\|_{w_1}^p + \|f\|_{w_2}^p$ on $W_{w_1}^p(IR)$. First we proved that $\left(W_{w_1}^p(IR), |||\cdot|||\right)$ is a Banach space. Let $(f_n)_{n\in N}$ be a Cauchy sequence in $\left(W_{w_1}^p(IR), |||\cdot|||\right)$. It is easy to see from the definition of $|||\cdot|||$ that the sequence $(f_n)_{n\in N}$ is also a Cauchy sequence in the spaces $\left(W_{w_1}^p(IR), \|\cdot\|_{w_1}^p\right)$ and $\left(W_{w_2}^p(IR), \|\cdot\|_{w_2}^p\right)$. Since $\left(W_{w_1}^p(IR), \|\cdot\|_{w_1}^p\right)$ and $\left(W_{w_2}^p(IR), \|\cdot\|_{w_2}^p\right)$ are Banach space, there exist $f \in W_{w_1}^p(IR)$ and $g \in W_{w_2}^p(IR)$ such that $\|f_n - f\|_{w_1}^p \to 0$ and $\|f_n - g\|_{w_2}^p \to 0$. Using the inequality $\|\cdot\|_1 \le \|\cdot\|_{W^p}$, we obtain $\|f_n - f\|_1 \to 0$ and $\|f_n - g\|_1 \to 0$. Therefore, there exists a subsequence $(f_{n_k}) \subset (f_n)$ such that $f_{n_k} \to f$ a.e. Also, there is a subsequence $(f_{n_{k_l}})$ of (f_{n_k}) such that $f_{n_{k_l}} \to g$ a.e. So we have $f = g$. Consequently, we obtain $|||f_n - f||| \to 0$. So $\left(W_{w_1}^p(IR), |||\cdot|||\right)$ is a Banach space. Now, let us define the unit function I

from $\left(W^p_{w_1}(IR), |||.|||\right)$ into $\left(W^p_{w_1}(IR), \|.\|^p_{w_1}\right)$. Since we have the inequality $\|I(f)\|^p_{w_1} = \|f\|^p_{w_1} \le |||f|||$, I is continuous. Hence I is a homeomorphism by the Banach Theorem[1]. That means the norms $|||.|||$ and $\|.\|^p_{w_1}$ are equivalent. Then for every $f \in W^p_{w_1}(IR)$ there exists $a > 0$ such that

$$|||f||| \le a\|f\|^p_{w_1}. \tag{2.7.1}$$

Thus by using (2.7.1) and the definition of norm $|||.|||$ we obtain

$$\|f\|^p_{w_2} \le |||f||| \le a\|f\|^p_{w_1}.$$

LEMMA 2.8 For any $f \in W^p_w(IR)$, $f \ne 0$ there exists a constant $c(\hat{f}) > 0$ such that

$$c(\hat{f})w(t) \le \|M_t f\|^p_w \le w(t)\|f\|^p_w .$$

Moreover, there is $c > 0$ such that

$$c\,w(t) \le \|M_t\|^p_w \le w(t) .$$

PROOF: Let $f \in W^p_w(IR)$, $f \ne 0$ be given. Then $\hat{f} \in L^q_w(IR)$ and there exists $c(\hat{f}) > 0$ such that

$$c(\hat{f})w(t) \le \left\|L_t\hat{f}\right\|_{q,w} \le w(t)\left\|\hat{f}\right\|_{q,w}$$

by the Lemma 2.2 in [2]. Hence we write

$$c(\hat{f})w(t) \le \|f\|_{W^p} + c(\hat{f})w(t) \le \|M_t f\|_{W^p} + \left\|L_t\hat{f}\right\|_{q,w} =$$

$$= \|M_t f\|_{W^p} + \left\|\widehat{M_t f}\right\|_{q,w} = \|M_t f\|^p_w. \tag{2.8.1}$$

On the other hand, one has

$$\|M_t f\|^p_w = \|M_t f\|_{W^p} + \left\|\widehat{M_t f}\right\|_{q,w} = \|f\|_{W^p} + \left\|L_t\hat{f}\right\|_{q,w} \le$$

$$\leq \|f\|_{W^p} + w(t)\left\|\hat{f}\right\|_{q,w} \leq w(t)\ \|f\|_w^p. \tag{2.8.2}$$

If one combines (2.8.1) and (2.8.2) obtains that

$$c(\hat{f})w(t) \leq \|M_t f\|_w^p \leq w(t)\|f\|_w^p .$$

THEOREM 2.9 Let w_1 . w_2 be weight functions on IR and $1 \leq p,q < \infty$. Then $W_{w_1}^p(IR) \subset W_{w_2}^p(IR)$ if and only if $w_2 \prec w_1$.

PROOF: Suppose that $W_{w_1}^p(IR) \subset W_{w_2}^p(IR)$. By the Lemma 2.8, there are constants $c > 0$ and $d > 0$ sucb that

$$c^{-1} w_1(t) \leq \|M_t f\|_{w_1}^p \leq c\, w_1(t) \tag{2.9.1}$$

and

$$d^{-1} w_2(t) \leq \|M_t f\|_{w_2}^p \leq d\, w_2(t) \tag{2.9.2}$$

for all $f \in W_{w_1}^p(IR)$ and $t \in IR$. Moreover, since $M_t f \in W_{w_1}^p(IR)$ for all $f \in W_{w_1}^p(IR)$, then by the Lemma 2.7, there exists constant $a > 0$ such that

$$\|M_t f\|_{w_2}^p \leq a\|M_t f\|_{w_1}^p . \tag{2.9.3}$$

Thus using (2.9.1), (2.9.2) and (2.9.3), we obtain

$$d^{-1} w_2(t) \leq \|M_t f\|_{w_2}^p \leq a\|M_t f\|_{w_1}^p \leq a\,c\, w_1(t) .$$

We set $k = a\,c\,d$, then $w_2(t) \leq w_1(t)$ for all $t \in IR$. That means $w_2 \prec w_1$.

Conversely, suppose that $w_2 \prec w_1$. Then there exists a constant $r > 0$ such that $w_2(t) \leq r\, w_1(t)$ for every $t \in IR$. Let $f \in W_{w_1}^p(IR)$ be given. Then we have $f \in W^p(IR)$ and $\hat{f} \in L_{w_1}^q(IR)$. Thus

$$\left\|\hat{f}\right\|_{q,w_2} = \left\{ \int_{IR} \left|\hat{f}(t) w_2(t)\right|^q dt \right\}^{1/q} \leq \left\{ \int_{IR} \left|\hat{f}(t) w_1(t)\right|^q dt \right\}^{1/q} = r\left\|\hat{f}\right\|_{q,w_1} < \infty .$$

Hence, we find $\hat{f} \in L^q_{w_2}(IR)$. Since $f \in W^p(IR)$, then we write $f \in W^p_{w_2}(IR)$. Consequently, we have $W^p_{w_1}(IR) \subset W^p_{w_2}(IR)$.

3.MULTIPLIERS FROM $L^1(IR)$ TO $W^p_w(IR)$

It is known that $L^1(IR)$ has an bounded approximate identity $(\hat{u}_\alpha)$ such that $\|u_\alpha\|_1 = 1$ and $\hat{u}_\alpha$ has a compact support for every α. We define the set $M_{W^p_w}$ is the following way:

$$M_{W^p_w} = \left\{ \mu \in M(IR) \middle| \; \|u_\alpha * \mu\|^p_w < c_\mu \right\}$$

where c_μ is a constant only depend on μ. Define the norm by

$$\|\mu\|_{M_{W^p_w}} = \overline{\lim_\alpha} \|u_\alpha * \mu\|^p_w .$$

Since $W^p_w(IR)$ is a Segal algebra by the Theorem 2.5, then the space of multipliers $M\left(L^1(IR), W^p_w(IR)\right)$ is the space $M_{W^p_w}$ by the Theorem 1.6 in [5].

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