İstanbul Üniv. Fen Fak. Mat. Dergisi 55 - 56 (1996 - 1997), 51-59

AN APPROXIMATE SOLUTION FOR THE ONE-DIMENSIONAL SINGULAR INTEGRAL EQUATIONS

S. M. AMER

Abstract

The present paper is concerned with the method of mechanical quadrature for the approximate solution of one - dimensional nonlinear singular integral equation in generalized Hölder space.

Introduction

The theory of approximation methods and its applications for the solution of linear and nonlinear singular integral equations has been developed by many authors, e.g. Mikhlin, S.G. and Prossdorf, S. [5], Peter Junghanns, et. al. [7]; Gakhov, F.D. [2], Guseinov, A.I. and Mukhtarov, Kh. Sh. [4], and others.

The purpose of this paper is to investigate the approximate solution of the following class of nonlinear singular integral equations (NSIE):

$$u(s) = \frac{\lambda}{2\pi} \int_0^{2\pi} F\,[u(\sigma)] \cot \frac{\sigma - s}{2}\, d\sigma\ , \tag{1}$$

in generalized Hölder spaces $H_{\phi\,,\,m}$ and $H^{(N)}_{\phi\,,\,m}$, [4,5,8].

1. The solution in the space $H_{\phi\,,\,m}$.

Definition 1.1. a) Let m be a natural number, the function $u(t)$ which is 2π - periodic in t and continuous belongs to the space $H_{\phi\,,\,m}$ iff $\omega_u^m(\delta) = 0(\phi(\delta))$, $0 < \delta \le \pi$, where $\omega_u^m(\delta)$ is the modulus of continuity of order m of u and ϕ is a positive non-decreasing function defined on $(0\,,\,\pi]$, for $t > 0$, such that $\lim\limits_{t \to 0^+} \phi(t) = 0$.

b) The formula $\| u \|_{\phi\,,\,m} = \max\limits_{x \in [-\pi\,,\,\pi]} | u(x) | + \sup\limits_{0 < \delta \le \pi} \dfrac{\omega_u^m(\delta)}{\phi(\delta)}$ defines a norm in the space $H_{\phi\,,\,m}$.

c) For $u \in H_{\phi\,,\,m}$ we define $H_{\phi,m}(M) = \left\{ u : \| u \|_{\phi,m} \le M\,,\, M > 0 \right\}$.

d) We say that ϕ belongs to the class Φ^m if

$$t_1^m\,\phi(t_2) \le c(m)\, t_2^m\,\phi(t_1)\,,\ \text{for } 0 < t_1 < t_2 < \pi\ .$$

Also, we define $H\Phi^m = \left\{ \phi : \int\limits_0^\delta t^{-1}\phi(t)\,dt + \delta^m \int\limits_\delta^\pi t^{-m-1}\phi(t)\,dt = 0\left(\phi(\delta)\right) \right\}$.

Lemma 1.2. Let the function $F[u]$ defined on $[-M\,,\,M]$, $(M > 0)$ and has $(m-1)$ derivatives and for arbitrary $u_1\,,u_2 \in [-M\,,M]$, the following condition is valid :

$$\left| F^{(q)}[u_1] - F^{(q)}[u_2] \right| \le \alpha(q) \left| u_1 - u_2 \right|,\ q = 0\,,\ldots,\,m-1 \qquad (1.1)$$

then, $\omega_F^m(\delta) \le \alpha(m)\ \omega_u^m(\delta)$, where $\alpha(m)$ is a constant .

Proof. For $m = 1$, the lemma is true. For $m = 2$, we have

$\Delta_h^2 F[u] = F\left[u(\sigma + 2h)\right] - 2F\left[u(\sigma + h)\right] + F\left[u(\sigma)\right]$. Hence,

$$\left|\Delta_h^2 F\right| = \left| \left\{F\left[u(\sigma + 2h)\right] - F\left[u(\sigma + h)\right]\right\} - \left\{F\left[u(\sigma + h)\right] - F\left[u(\sigma)\right]\right\} \right| .$$

Using Lagrange's formula, we have

$$\left|\Delta_h^2 F\right| = \left| \int\limits_0^1 F_u'\left[u(\sigma + h) + \theta\left(u(\sigma + 2h) - u(\sigma + h)\right)\right]\left[u(\sigma + 2h) - \right.\right.$$

$$\left. - u(\sigma + h)\right] d\theta - \int\limits_0^1 F_u'\left[u(\sigma) + \theta\left(u(\sigma + h) - u(\sigma)\right)\right]\left[u(\sigma + h) - \right.$$

$$- u(\sigma)]\, d\theta \,| = | \int_0^1 \{F'_u [u(\sigma + h) + \theta (u(\sigma + 2h) - u(\sigma + h))] -$$

$$- F'_u [u(\sigma) + \theta (u(\sigma + h - u(\sigma))]\} [u(\sigma + 2h) - u(\sigma + h)]\, d\theta +$$

$$+ \int_0^1 F'_u \,[u(\sigma) + \theta (u(\sigma + h) - u(\sigma))]\, [u(\sigma + 2h) - 2u(\sigma + h) +$$

$$+ u(\sigma)]\, d\theta \,\Big|, \qquad 0 \le \theta \le 1 \,.$$

Applying condition (1.1), we get

$$\left| F'_u \,[u(\sigma) + \theta\, (u(\sigma + h) - u(\sigma))]\right| \le \alpha\,(1)\, M + \Omega_{1'}$$

where $\left| F_u^{(m-1)'}(0) \right| \le \Omega_{m-1'}$

then, $\left|\Delta_h^2 F\right| \le \alpha\,(1)\left[\omega_u(h) + \omega_u^2(h)\right]\omega_u(h) + \left(\alpha\,(1)\, M + \Omega_1\right)\omega_u^2(h)$.

Consequently, we have $\omega_F^2(\delta) \le \text{const.}\ \omega_u^2(\delta)$, then the lemma is true at $m = 2$. By induction, we see that

$$\Delta_h^m\, F[u(\sigma)] = \sum_{\mu=0}^{m-1} \binom{m-1}{\mu} \int_0^1 \Delta_h^\mu F'_\mu [u(\sigma) + \theta\, (u(\sigma + h)$$

$$- u(\sigma))]\ \Delta_h^{m-\mu} u(\sigma + \mu h)\, d\theta \,.$$

The function $F'_u\, [u(\sigma) + \theta\, (u(\sigma + h) - u(\sigma))]$ has $(\mu - 1)$- derivatives, $\mu = 1, \ldots, m - 1$, and these derivatives satisfy condition (1.1), then we obtain $\left|\Delta_h^m F[u(\sigma)]\right| \le \text{const.}\ \omega_u^\mu(\delta)\ \omega_u^{m-\mu}(\delta) \le \text{const.}\ \omega_u^\mu(\delta)$, that is, $\omega_F^m(\delta) \le \alpha\,(m)\, \omega_u^m(\delta)$. Thus the lemma is proved.

Remark. From definition 1.1 and lemma 1.2, it is clear that if $u(\sigma) \in H_{\phi, m}$ then $F[u(\sigma)] \in H_{\phi, m}$.

Theorem 1.3 [6]. Let $\phi \in H\Phi^m$, then the operator

$$(A\,u)\,(x) = \widetilde{u}\,(x) = \frac{1}{2\pi} \int_0^{2\pi} u(y) \cot \frac{y - x}{2}\, dy \ ,$$

transforms $H_{\phi,m}(M)$ into $H_{\phi,m}(\widetilde{M})$ where

$$\widetilde{M} = M\left[c_1(m)\int_0^{\pi}\frac{\phi(\zeta)}{\zeta}\,d\zeta + c_1(m) + c_2(m)\,\widetilde{c}(m)\right] .$$

Theorem 1.4. If the function $F[u(\sigma)]$ satisfy the condition (1.1), then for $|\lambda| < |\lambda_0|$ (λ_0 sufficiently small), equation (1) has a unique solution in $H_{\phi,m}(M)$. This solution is uniformly convergent and can be obtained by the method of successive approximations .

Proof. Let $u \in H_{\phi,m}(M)$, then by lemma 1.2 and theorem 1.3, the operator $(Au)(s) = \frac{\lambda}{2\pi}\int_0^{2\pi} F[u(\sigma)] \cot\frac{\sigma - s}{2}\,d\sigma$ maps $H_{\phi,m}(M)$ into $H_{\phi,m}(|\lambda| D)$. Hence, if $|\lambda| D \le M$, the operator (Au) maps $H_{\phi,m}(M)$ into itself for som $D > 0$. On using M. Riesz's theorem [9], we have $\|\widetilde{u}\|_{L_P} \le \eta(P)\,\|u\|_{L_P}$, $1 < P < \infty$

where $\widetilde{u}(s) = \frac{1}{2\pi}\int_0^{2\pi} u(\sigma)\cot\frac{\sigma - s}{2}\,d\sigma$.

Now, $$\|Au_1 - Au_2\|_{L_P} = \left\{\int_0^{2\pi}\left|\frac{\lambda}{2\pi}\int_0^{2\pi}\{F[u_1(\sigma)] - F[u_2(\sigma)]\}\times\right.\right.$$

$$\left.\left.\times\cot\frac{\sigma - s}{2}\,d\sigma\right|^P ds\right\}^{1/P} \le |\lambda|\,\eta(P)\,\|F[u_1(\sigma)] - F[u_2(\sigma)]\|_{L_P} =$$

$$= |\lambda|\,\eta(P)\left\{\int_0^{2\pi}|F[u_1(\sigma)] - F[u_2(\sigma)]|^P\,d\sigma\right\}^{1/P} \le$$

$$\le |\lambda|\,\eta(P)\,\alpha(0)\,\|u_1 - u_2\|_{L_P} .$$

If $|\lambda|\,\eta(P)\,\alpha(0) < 1$, then the operator A is a contraction mapping;

$$|\lambda| < |\lambda_0| = \min\left\{\frac{M}{D}, \frac{1}{\eta(P)\,\alpha(0)}\right\} .$$

By completeness of $H_{\phi,m}(M)$ in L_P, $1 < P < \infty$, shown below, the equation (1) has a unique solution $H_{\phi,m}(M)$ and it can be evaluated by successive approximations .

Theorem 1.5. Let $u_n , u \in H_{\phi,m}(M)$ and $\lim\limits_{n \to \infty} \| u_n - u \|_{L_P} = 0$, then, $\lim\limits_{n \to \infty} \| u_n - u \|_c = 0$.

Proof. Let $G \in H_{\phi,m}(M)$ can be written in the form

$$G(s) = \frac{1}{h} \int_s^{s+h} G(x)\, dx - \frac{1}{h} \int_s^{s+h} [G(x) - G(s)]\, dx ,$$

then we have $|G(s)| \le \frac{1}{h} \int_s^{s+h} |G(x)|\, dx + \frac{1}{h} \int_s^{s+h} |G(x) - G(s)|\, dx$.

Using Hölder inequality on the first term in the right part of the last inequality, we have

$$\|G(s)\|_c \le \frac{1}{h} \left(\int_s^{s+h} |G(x)|^P dx \right)^{1/P} h^{1/q} + \frac{1}{h} \int_s^{s+h} \|G\|_\phi \, \phi(|x-s|)\, dx \le$$

$$\le \left(h^{1-1/q}\right)^{-1} \left(\int_{-\pi}^{\pi} |G(x)|^P dx \right)^{1/P} + \|G\|_\phi \, \phi(h) \ .$$

Putting $G(s) = u_n(s) - u(s)$, $h = \|u_n - u\|_{L_P}$,

we obtain $\|u_n - u\|_c \le \|u_n - u\|_{L_P}^{1/q} + 2M\, \phi\left(\|u_n - u\|_{L_P}\right)$,

then $\|u_n - u\|_c \to 0$ as $n \to \infty$, that is the successive approximations converges .

2. The solution in the discrete space $H_{\phi,m}^{(N)}$

Definition 2.1. Denote to $H_{\phi,m}^{(N)}$, $\phi \in \Phi^m$, be the 2N-dimensional space of vectors z ; $z = \left(z_0, z_1, \ldots, z_{2N-1}\right)$, with the norm :

$$\| z \|_{\phi,m}^{(N)} = \max \left\{ \max_{k=0,1,\ldots 2N-1} , \max_{h>0} \frac{\omega^m (z , h)}{\phi\left(\frac{\pi h}{N}\right)} \right\} .$$

Also, we define:

$$H_{\phi,m}^{(N)}(M) = \left\{ z \in H_{\phi,m}^{(N)} : \| z \|_{\phi,m}^{(N)} \leq M \right\} \text{, as a subspace of } H_{\phi,m}^{(N)} .$$

For singular integral

$$(Ju)(s) = \frac{1}{2\pi} \int_0^{2\pi} u(\sigma) \cot \frac{\sigma - s}{2} d\sigma \ , \ u(\sigma) \in H_{\phi,m} \ ,$$

the quadrature formula at the node points $s_k = \frac{k\pi}{N}$ takes the following form; [3],

$$(Ju)(s_k) \approx \frac{1}{2N} \sum_{\substack{k=0 \\ k \neq t}}^{2N-1} u_k \left[1 - (-1)^{k-t}\right] \cot \frac{s_k - s_t}{2} \ . \qquad (2.1)$$

Applying the quadrature formula (2.1) to equation (1), we obtain

$$u(s_\gamma) = \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k \neq \gamma}}^{2N-1} F[u(s_k)] \left[1 - (-1)^{k-\gamma}\right] \cot \frac{s_k - s_\gamma}{2} + R_N[F] \ ,$$

where $R_N[F]$ is the remainder term, $\gamma = 0, \ldots, 2N-1$. Put $u(s_\gamma) = z_\gamma$, we get the following system of nonlinear algebraic equations (SNAE):

$$z_\gamma = \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k \neq \gamma}}^{2N-1} F[z_k)] \left[1 - (-1)^{k-\gamma}\right] \cot \frac{s_k - s_\gamma}{2} \qquad (2.2)$$

Theorem 2.2. Let the function $F[u]$ satisfies condition (1.1), then, for arbitrary $N \geq 3$ and sufficiently small $|\lambda|$, the SNAE (2.2) , has unique solution in $H_{\phi,m}^{(N)}(M)$ and this solution can be found by successive approximations.

Proof. Let $z = (z_k) \in H_{\phi,m}^{(N)}(M)$, $Hz = (F(z_k))$, $k = 0, \ldots, 2N-1$, and since the space $H_{\phi,m}^{(N)}(M)$ is a closed subspace of $L_p^{(N)}$, and the function $F(z)$ satisfy the conditions of lemma 1.2 and theorem 1.4. Then we obtain $Hz \in H_{\phi,m}^{(N)}(D')$. Let $G^{(N)}z = \lambda\, A^{(N)}\, Hz$,

where $A^{(N)}z = \left(A_0^{(N)}z\,,\, A_1^{(N)}z\,,\dots,\, A_{2N-1}^{(N)}z\right)$, and $\left\| A^{(N)} \right\|_{\phi,m}^{(N)} \le c(m)$,

(see [8]). Hence $\left\| G^{(N)}z \right\|_{\phi,m}^{(N)} \le |\lambda|\, D'\, c(m)$.

Since $\left\| G^{(N)}z \right\|_{L_P^{(N)}} \le c(P)\,,\, 1 < P < \infty$, (see [8]). Using condition (1.1), we get

$$\left\| G^{(N)}z^{(1)} - G^{(N)}z^{(2)} \right\|_{L_P^{(N)}} \le |\lambda|\, c(P)\, \alpha(0) \left\| z^{(1)} - z^{(2)} \right\|_{L_P^{(N)}} . \quad (2.3)$$

Using the contraction mapping principle at

$$|\lambda| < \min\left\{ \frac{M}{D'\, c(m)}, \frac{1}{c(P)\, \alpha(0)} \right\},$$

the SNAE (2.2) has a unique solution in $H_{\phi,m}^{(N)}(M)$, for arbitrary $N \ge 3$, and theorem is proved .

3. The rate of convergence of the approximate solution

If $$|\lambda| < \min\left\{ \frac{M}{D'\, c(m)}, \frac{1}{c(P)\, \alpha(0)} \right\}, \quad (3.1)$$

then the equation (1) has a unique solution $u^*(\sigma) \in H_{\phi,m}(M)$ and the system (2.2), for arbitrary $N \ge 3$, has a unique solution

$$z^* = \left(z_0^*\,,\, z_1^*\,,\dots,\, z_{2N-1}^*\right) \in H_{\phi,m}^{(N)}(M) .$$

The approximate solution of equation (1) takes the form

$$u^{(N)}(s) = \frac{\lambda}{N} \sum_{k=0}^{2N-1} F\left[z_k^*\right] \sin^2\frac{s - s_k}{2} \cot\frac{s_k - s}{2} , \quad (3.2)$$

at $s = s_\gamma$ (k - different from γ simultaneously) .

Applying the quadrature formula (2.1) to equation (1) at node points s_γ , we obtain

$$u^*(s_\gamma)=\frac{\lambda}{2N}\sum_{\substack{k=0\\k\neq\gamma}}^{2N-1}F\left[u^*(s_k)\right]\left[1-(-1)^{k-\gamma}\right]\cot\frac{s_k-s_\gamma}{2}+R_N[F] \tag{3.3}$$

Put $z^{(1)}=u^*$ and $z^{(2)}=z^*$ in (2.3) and from (3.1), we get

$$\left\|u^*-z^*\right\|_{L_P^{(N)}}\leq|\lambda|\left\|R_N[F]\right\|_c\left\{1-|\lambda|\ c(p)\,\alpha\,(0)\right\}^{-1} \ . \tag{3.4}$$

From [3], we get

$$u^*(s)-u^{(N)}(s)=\frac{\lambda}{N}\sum_{k=0}^{2N-1}\left\{F\left[u^*(s_k)\right]-F\left[z_k^*\right]\right\}\sin^2\frac{s-s_k}{2}\cot\frac{s_k-s}{2}+$$
$$+|\lambda|R_N[F] \ . \tag{3.5}$$

From [1] and condition (1.1), we get

$$\left\|u^*(s)-u^{(N)}(s)\right\|_c\leq 2|\lambda|\,\alpha(0)\,(1+\pi)\,(1+\ln 2N)\max_\gamma\left|u^*(s_\gamma)-z_\gamma^*\right|+$$
$$+|\lambda|\left\|R_N[F]\right\|_c \ . \tag{3.6}$$

From [6];

$$\max_\gamma\left|u^*(s_\gamma)-z_\gamma^*\right|\leq \text{const.}\left[\left(\frac{N}{h}\right)^{1/P}\left\|u^*-z^*\right\|_{L_P^{(N)}}+\phi\left(\frac{\pi h}{N}\right)\right] \ ,$$
$$0<h<\frac{N}{2\,(m+1)} \tag{3.7}$$

and $\left\|R_N[F]\right\|_c\leq c(m)\ \phi\left(\frac{\pi}{N}\right)\ln N$.

From (3.4) - (3.7), we get

$$\left\|u^*(s)-u^{(N)}(s)\right\|_c\leq \text{const.}\left(\frac{\phi\left(\frac{\pi}{N}\right)\ln^2 N}{N^{(t-1)/P}}\right),\ t>1 \text{ and } 1<P<\infty \ .$$

References

[1] Babaev, A.A., Malsagov, S.M. and Salaev, V.V.; Basis of quadratic method for nonlinear singular integral equation with Hilbert kernel (in Russian), Ych. Zap. AGU. Bra. Nauk, 1, 13-33, (1971).

[2] Gakov, F.D.; Boundary value problems; Pergamon Press Ltd. (1971).

[3] Gabdulkhaev, B.G. and Duskov, P.N.; The method of mechanical quadratures for singular integral equations; Izv. Vyss. Ucebn Zaved Mtematika, 12, 3-14 (1974).

[4] Guseinov, A.I. and Mukhtarov, Kh.Sh.; Introduction to theory of nonlinear singular integral equations (in Russian), Nauk Moscow, (1980).

[5] Mikhlin, S.G. and Prossdorf, S.; Singular integral operators; Springer - Verlag Berlin (1986).

[6] Mosaev, B.I. and Salaev, V.V.; On the convergence of quadratic process for singular integrals with Hilbert Kernel; (in Russian), In Timely problems in the theory of functions; P. 186-194, AUG. (1980).

[7] Peter Junghanns, Capobianco M.R., Luther U. and Mastroianni G.; Weighted uniform convergence of the Quadrature Method for cauchy singular integral equations, operator theory advances and applications, Vol. 90, Birkhauser Verlag (1996) 153-181.

[8] Saleh, M.H. and Amer S.m.; Singular operator in generalized Holder spaces; Proc. Phys. Soc. Egypt No. 64, pp. 91-112 (1987).

[9] Zygmund, A., Trigonometric series; Cambridge University Press, Cambridge (1959).

Department of Mathematics,
Faculty of Science
Zagazig University
Egypt