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ON SOME AFFINE CONNECTIONS ON MANIFOLDS WITH ALMOST CONTACT 3-STRUCTURE

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Abstract. In this paper, we shall study affine connections on manifolds with almost contact structure and with almost contact 3-structure. Using Obata's operators associated to an almost contact structure (φ, ξ, η) or a three almost contact structures $(\varphi_i, \xi_i, \eta^i)$ $(i = 1,2,3)$ and Wilde's method of characterizing the set of solutions of a system of tensorial equations are found all the (φ, ξ, η)-affine connections and $(\varphi_i, \xi_i, \eta^i)$-affine connections and their groups of transformations.

INTRODUCTION

Let M be a $(2n+1)$-dimensional C^∞ manifold and let $\Im(M)$ be the algebra of all the differentiable functions on M. We denote by $T_s^r(M)$ the $\Im(M)$-module of the tensor fields of type (r,s). For $T_0^1(M)$ is used the notation $\aleph(M)$. Let $C(M)$ be the affine modul of affine connections on M.

An almost affine contact structure on M is defined by a C^∞ $(1,1)$-tensor field φ, a C^∞ vector field ξ and a C^∞ one-form η on M such that

$$(1) \qquad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

where $\otimes$ denotes the tensor product and I is the identity tensor. This implies $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Manifolds equipped with an almost contact structure are called almost contact manifolds.[1]

Let us suppose that a differentiable manifold admits three almost contact structures $(\varphi_i, \xi_i, \eta^i)$ $(i = 1,2,3)$ satisfying

$$(2) \qquad \begin{aligned} &\varphi_i\varphi_j - \eta^j \otimes \xi_i = -\varphi_j\varphi_i + \eta^i \otimes \xi_j = \varphi_k, \quad \eta^i(\xi_j) = \delta_j^i, \\ &\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \quad \eta^i \circ \varphi_j = -\eta^j \circ \varphi_i = \eta^k \end{aligned}$$

for any cyclic permutation (i,j,k) of $(1,2,3)$. Then $(\varphi_i, \xi_i, \eta^i)$ $(i = 1,2,3)$ is called an almost contact 3-structure.[3]

1. Affine connections on almost contact manifolds

Let M be a differentiable manifold with an almost contact structure (φ, ξ, η). We consider the distribution $H = Ker\,\eta$ and $V = Ker\ \varphi = \{\xi\}$ on M and we denote

(3) $$h = I - \xi \otimes \eta,\ v = \xi \otimes \eta$$

the projections on H and V respectively. We have [2] ,

(4) $$h^2 = h,\ v^2 = v,\ hv = vh = 0$$
$$\varphi^2 = -h,\ h\varphi = \varphi h = \varphi,\ v\varphi = \varphi v = 0$$

Thus h and v are complementary projection operators on M.

Definition 1.1. We call Obata operators associated to an almost contact structure (φ, ξ, η), the applications $A, A^* : T_1{}^1(M) \to T_1{}^1(M)$ defined by

(5) $$A(w) = v \circ w \circ v + h \circ w \circ h,\ A(w) = v \circ w \circ h + h \circ w \circ v$$

Proposition 1.1. A and A^* are complementary projection operators on $T_1{}^1(M)$.

Proposition 1.2. The tensorial equation

(6) $$A^*(u)\ a,\ \ a \in T_1{}^1(M)$$

has a solution $u \in T_1{}^1(M)$ if and only if $a \in Ker\,A$. If $a \in Ker\,A$, then the general solution of the equation (6) is

(6') $$u = a + A(w),\ \forall\ w \in T_1{}^1(M).$$

A similar result holds for the equations of the form $A(u) = a$.

In the following, $\overset{\circ}{\nabla} \in C(M)$ will be an affine connection fixed on M such that, $\overset{\circ}{\nabla} \xi = 0,\ \overset{\circ}{\nabla} \eta = 0$. Every tensor field $u \in T_1{}^1(M)$ may be considered as a field of $\aleph(M)$–valued differential 1-forms. So, if ∇ is an affine connection on M, then we note with D and $\widetilde{D}$ the associated connections acting on the $\aleph(M)$-valued differential 1-forms and respectively on the differential 1- forms:

(7) $$(D_X u)Y = \nabla_X(uY) - u(\nabla_X Y)$$

(8) $$(\widetilde{D}_X \eta)Y = X(\eta(Y)) - \eta(\nabla_X Y)$$
$\forall\ u \in T_1{}^1(M)$ and $X, Y \in \aleph(M)$.

Definition1.2. An affine connection ∇ on M is called an (φ,ξ,η)– affine connection if

(9) $\quad D\varphi = 0,\ \widetilde{D}\eta = 0,\ \nabla\xi = 0.$

Of course, for every (φ,ξ,η)–affine connection ∇, we have

(10) $\quad \nabla_X v = v\nabla_X,\ \nabla_X h = h\nabla_X;\quad \forall X \in \aleph(M)$

We see that D and $\widetilde{D}$ commute with the operators A and A^*.

We take $\nabla_X = \overset{\circ}{\nabla}_X + V_X$, where $\overset{\circ}{\nabla}$ is an affine connection on M such that $\overset{\circ}{\nabla}\xi = 0,\ \overset{\circ}{D}\eta = 0$ and $V_X Y = V(X,Y),\ V \in T_2^{\,1}(M)\ \forall\ X, Y \in \aleph(M)$ and we find the tensor field V so that it satisfies the conditions (9).

∇ will be an (φ,ξ,η)-affine connection if and only if the field V satisfies the system of the tensorial equations:

(11) $\quad V_X \circ \varphi - \varphi \circ V_X = -(\overset{\circ}{D}_X\varphi),\quad \forall X \in \aleph(M)$

(12) $\quad \eta \circ V_X = 0,\quad V_X\xi = 0,\quad \forall X \in \aleph(M)$

We have also

$$V_X \circ v - v \circ V_X = -\overset{\circ}{D}_X v$$

(13)

$$V_X \circ h - h \circ V_X = -\overset{\circ}{D}_X h$$

which implies that

$$h \circ V_X \circ v = -h \circ \overset{\circ}{D}_X v = (\overset{\circ}{D}_X h)\circ v$$

(14)

$$v \circ V_X \circ v = -v \circ \overset{\circ}{D}_X v = (\overset{\circ}{D}_X v)\circ h$$

$\forall X \in \aleph(M)$. Putting

(15) $\quad a(X) = (\overset{\circ}{D}_X h)\circ v + (\overset{\circ}{D}_X v)\circ h = -h\circ(\overset{\circ}{D}_X h) - v\circ(\overset{\circ}{D}_X h)$

it follows that V_X must verify the system

(16) $\quad A^*(V_X) = a(X),\quad \eta\circ V_X = 0,\quad V_X\xi = 0.$

But

$$A(a(X)) = A((\overset{\circ}{D}_X h)\circ v + (\overset{\circ}{D}_X v)\circ h) = v\circ(h\circ \overset{\circ}{D}_X v + v\circ \overset{\circ}{D}_X h)\circ v +$$
$$h\circ(h\circ \overset{\circ}{D}_X v + v\circ \overset{\circ}{D}_X h)\circ h = v\circ \overset{\circ}{D}_X h\circ v + h\circ \overset{\circ}{D}_X v\circ h = 0,$$

and by a straightforward computation, it is verified

$$\eta \circ a(X) = 0, \quad a(X)\xi = 0, \quad \forall X \in \aleph(M).$$

Applying the Proposition 1.2, it becomes that the system (16) has a solution and the general solution is

$$V_X = a(X) + A(W_X), \tag{17}$$

where $W \in T_2^{\,1}(M)$ must be verify the conditions

$$\eta \circ A(W_X) = 0, \; A(W_X)(\xi) = 0, \quad \forall X \in \aleph(M) \tag{18}$$

Then we obtain the following:

Theorem1.1. There are (φ, ξ, η)–affine connections: one of them is

$$\nabla_X = \overset{\circ}{\nabla}_X + (\overset{\circ}{D}_X h) \circ v + (\overset{\circ}{D}_X v) \circ h \tag{19}$$

where $\overset{\circ}{\nabla}$ is an affine connection on M such that $\overset{\circ}{\nabla} \xi = 0$ and $\bar{D} \eta = 0$, $\overset{\circ}{D}$ and $\tilde{D}$ being its associate connections.

Theorem1.2. The set of all (φ, ξ, η)– affine connections is given by

$$\overline{\nabla}_X = \nabla_X + A(W_X) \tag{20}$$

where ∇ is an (φ, ξ, η) -affine connection and $W \in T_2^{\,1}(M)$ satisfies the conditions (18).

Observing that (18) and (20) can be considered as a transformation of (φ, ξ, η)– affine connections, we have :

Theorem1.3. The set of the trnsfornations of (φ, ξ, η)– affine connections and the multiplication of the applications is an abelian group, noted with $G(\varphi, \xi, \eta)$, isomorph with the additive group of the tensor $W \in T_2^{\,1}(M)$ which satisfies the conditions (18) and (20).

2.Affine connections on manifolds with an almost contact 3-structure

Let M be a differentiable manifold with an almost contact 3- structure(φ , ξ i, η^i) $(i = 1,2,3)$. Now we consider the distributions $H_i = Ker\,\eta^i$ and $V_i = Ker\,\varphi_i$ and we denote

(21) $\quad h_i = I - \xi_i \otimes \eta^i, \quad v_i = \xi_i \otimes \eta^i, \; i = 1,2,3$

the projections on H_i and V_i respectively. We have

$h^2_i = h_i,\; v^2_i = v_i,\; h_i v_i = v_i h_i = 0,$
$\varphi_i^2 = - h_i,\; h_i\,\varphi_i = \varphi_i\, h_i = \varphi_i,\quad v_i\,\varphi_i = \varphi_i\, v_i = 0$
$v_i\, v_j = v_j\, v_i = 0,\; h_i\, v_j = v_j h_i = v_j$
$h_i\, h_j = h_j\, h_i = I - v_i - v_j,$ for $i \neq j$.

Definition 2.1. We call Obata operators associated to an almost contact 3-structure , $(\varphi_i, \xi i, \eta^i)$ $(i = 1,2,3)$ the applications A_i, $A^*_i : T_1{}^1(M) \to T_1{}^1(M)$ defined by

(23) $\quad A_i(w) = v_i \circ w \circ v_i + h_i \circ w \circ h_i,\; A_i(w) = v_i \circ w \circ h_i + h_i \circ w \circ v_i.$

Proposition 2.1. For an $(\varphi_i, \xi i, \eta^i)$ –structure on M and A_i, A^*_i defined by (23) we have

1) A_i and A^*_i are complementary projection operators on $T_1{}^1(M)$
2) A_i and A^*_i commute pairwise with A_j and A^*_j, $i \neq j$
3) $A_i \circ Aj$ and $A_i{}^* \circ A^*_j$ are projections on $T_1{}^1(M)$
4) $Ker\,A_i \cap Ker\,A_j = Im\,(A_i \cap A_j),\; i \neq j.$

Proof. 2) In fact, by simple calculation we have

$A_i \circ Aj\,(w) = v_i \circ Aj\,(w) \circ v_i + h_i \circ A_j(w) \circ h_i = v_i \circ v_j \circ w \circ v_j \circ v_i + v_i \circ h_j \circ w \circ hj \circ v_i + h_i \circ v_j \circ w \circ v_j \circ h_i + h_i \circ h_j \circ w \circ h_j \circ h_i = v_i \circ w \circ v_i + v_j \circ w \circ v_j + h_i \circ h_j \circ w \circ h_j \circ h_i.$

Proposition 2.2. The system of tensorial equations

(24) $\quad A^*_i(u) = a_i,\; i = 1,2,3$

has a solution $u \in T_1{}^1(M)$, if and only if

(25) $\quad A_i(a_i) = 0$ and $A_i(a_j) = Aj(a_i),\; i \neq j$

If the conditions (25) are fulfilled , then the general solution of the system (24) is

(26) $u = a_1 + A_1(a_2) + A_1A_2(a_3) + A_1A_2A_3(w)$

$\forall\ w \in \Im_1^1(M)$.

A similar result holds for the system of the form $A_i(u) = a_i$.

Definition 2.2. An affine connection ∇ on M is called an $(\varphi_i, \xi i, \eta^i)$–affine connection if

(27) $$D\varphi_i = 0,\ \tilde{D}\eta^i = 0,\ \nabla\xi_i = 0,\ i = 1,2,3.$$

For every $(\varphi_i, \xi i, \eta^i)$–affine connection ∇ we have

(28) $$\nabla_X v_i = v_i \nabla_X,\ \ \nabla_X h_i = h_i \nabla_X\ \ \forall X \in \aleph(M)$$

and D and $\tilde{D}$ commute with the operators A_i and A_i^*.

We take $\dot{\nabla}_X = \overset{\circ}{\nabla}_X + V_X$ where $\overset{\circ}{\nabla}$ will be an affine connection fixed on M such that $\overset{\circ}{\nabla}\xi i = 0$, $\overset{\circ}{\tilde{D}}\eta^i = 0$ $(i = 1,2,3)$ and $V_xY = V(X,Y)$, $\forall\ X,Y \in \aleph(M)$. ∇ will be an $(\varphi_i, \xi i, \eta^i)$–affine connection if and only if the field V satisfies the system of the tensorial equations

$$Ai^*(V_X) = a_i(X),\ \ \eta^i \circ V_X = 0,$$

(29)

$$V_X\xi_i = 0,\ i = 1,2,3,\ \forall X \in \aleph(M).$$

where

(30) $$a_i(X) = (\overset{\circ}{D}_X h_i) \circ v_i + (\overset{\circ}{D}_X v_i) \circ h_i = -h_i \circ \overset{\circ}{D}_X v_i - v_i \circ \overset{\circ}{D}_X h_i.$$

By a straightforward computation, it is proved that

(31) $$A^*_i(a_j(X)) = A^*_j(a_i(X))$$

is equivalent to

(32) $$h_i \circ \overset{\circ}{\nabla}_X h_j = h_j \circ \overset{\circ}{\nabla}_X h_i,\ \forall\ i \neq j.$$

Also, it is proved that

(33) $$\eta^i \circ a_j(X) = 0,\ a_i(X)(\xi_j) = 0,\ \forall\, i,j,\ \forall\ X \in \aleph(M)$$

If the conditions (32) and (33) are fulfilled , then the system (29) has nontrivial solutions and its general solution is given by

(34) $$V_X = a_1(X) + A_1a_2(X) + A_1A_2a_3(X) + A_1A_2A_3W_X$$

where $W \in T_2^1(M)$ must verify the conditions

(35) $\eta^{i} \circ A_1A_2A_3W_X = 0, \ A_1A_2A_3W_X\xi i = 0, \ i = 1,2,3, \ X \in \aleph(M).$

We have:

Theorem 2.1. There are $(\varphi_i, \xi i, \eta^i)$–affine connections: one of them is

(35) $\nabla_X = \overset{\circ}{\nabla}_X + a_1(X) + A_1a_2(X) + A_1A_2a_3(X)$

and the set of all $(\varphi_i, \xi i, \eta^i)$– affine connections is given by

(36) $\overline{\nabla}_X = \nabla_X + A_1A_2A_3(W_X)$

where ∇ is an $(\varphi_i, \xi i, \eta^i)$– affine connection and $W \in T_2^1(M)$ satisfies the conditions(35).

Observing that (35) and (37) can be considered as a transformation of $(\varphi_i, \xi i, \eta^i)$—affine connections, we have:

Theorem2.2. The set of all the transformations of $(\varphi_i, \xi i, \eta^i)$-affine connections and the multiplication of the applications is an abelian group, noted with $G(\varphi_i, \xi i, \eta^i)$–isomorph with the additive group of the tensors $W \in T_2^1(M)$ which satisfies the conditions (35) and (37).

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