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BOUNDEDNESS AND STABILITY RESULTS FOR A CERTAIN SYSTEM OF FIFTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract : *The paper studies equation (1.1) in two cases : (i) $P\equiv 0$, (ii) $P(\neq 0)$ satisfies, $\|P(t,X,Y,Z,W,U)\| \leq [\delta_1 + \delta_2 (\|Y\|+\|Z\|+\|W\|+\|U\|)]\theta(t)$, where $\theta(t)$ is a nonnegative function of t. In case (i) the asymptotic stability in the large of the trivial solution of (1.1) is investigated ; in case (ii) a boundedness result is deduced for solutions of (1.1). These results generalize some of the results obtained earlier.*

1. Introduction and statement of the results

We shall consider the real non-linear autonomous vector differential equation of fifth order

$$X^{(5)}+F(X,\dot X,\ddot X,\dddot X)X^{(4)}+\Phi(\dddot X,\ddot X)+G(\dot X,\ddot X)+H(\dot X)+\Psi(X)=P(t,$$

$$X,\dot X,\ddot X,\dddot X,X^{(4)}) \qquad \textbf{(1.1)}$$

or its equivalent system

$$\dot X = Y,\ \dot Y = Z,\ \dot Z = W,\ \dot W = U,$$
$$\dot U = -F(X,Y,Z,W)U - \Phi(Z,W) - G(Y,Z) - H(Y) - \Psi(X)+P(t,X,Y,Z,W,U) \qquad \textbf{(1.2)}$$

where $X \in R^n$, R^n denotes the real n-dimensional Euclidean space $R\times R\times\ldots\times R$ (n factors), $R=(-\infty,\infty)$, F is an $n\times n$ - matrix function, $\Phi : R^n \times R^n \to R^n$, $G : R^n \times R^n \to R^n$, $H : R^n \to R^n$, $\Psi : R^n \to R^n$ and $P : R\times R^n \times R^n \times R^n \times R^n \times R^n \to R^n$.

The dots as usual indicate differentiation with respect to t. The non -linear functions F, Φ, G, H and Ψ are continuous and so constructed such that the uniqueness theorem is valid. The equation (1.1) represents a system of real fifth-order differential equations of the form :

$$x_i^{(5)} + \sum_{k=1}^{n} f_{ik}(x_1,\ldots,x_n;\dot x_1,\ldots,\dot x_n;\ddot x_1,\ldots,\ddot x_n;\dddot x_1,\ldots,\dddot x_n)x_k^{(4)} + \phi_i(\ddot x_1,\ldots,\ddot x_n;\dddot x_1,\ldots,\dddot x_n)$$
$$+ g(\dot x_1,\ldots,\dot x_n;\ddot x_1,\ldots,\ddot x_n) + h_i(\dot x_1,\ldots,\dot x_n) + \psi_i(x_1,\ldots,x_n)$$
$$= p_i(x_1,\ldots,x_n;\dot x_1,\ldots,\dot x_n;\ddot x_1,\ldots,\ddot x_n;\dddot x_1,\ldots,\dddot x_n;x_1^{(4)},\ldots,x_n^{(4)}),\quad (i = 1,2,\ldots,n). \qquad \textbf{(1.3)}$$

Key words : System of non-linear differential equations of the fifth order, Lyapunov function, Global stability.

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Finding sufficient conditions for stability and boundedness of solutions is a problem of general interest in theory of differential equations. The problem in this paper, in the case $n=1$, has been the subject of intensive investigation in recent years (see, in particular, Abou-El-Ela & Sadek [3], Chukwu [5], Tunç ([10], [12]), Yuanhong [14]). Recently, Abou-El-Ela & Sadek [4] and Sadek [8] established sufficient conditions for the asymptotic stability in the large of the zero solution of the vector differentials equations

$$X^{(5)}+AX^{(4)}+\Phi(\ddot{X})+G(\ddot{X})+H(\dot{X})+BX=0$$

and

$$X^{(5)}+F(\ddot{X})X^{(4)}+\Phi(\ddot{X})+G(\ddot{X})+H(\dot{X})+\Psi(X)=0,$$

respectively.

Tunç [13] derived similar results for the problem

$$X^{(5)}+F(\ddot{X},\dddot{X})X^{(4)}+\Phi(\ddot{X},\dddot{X})+G(\ddot{X})+H(\dot{X})+\Psi(X)=0$$

The first result obtained is comparable in generality with the results of Abou-El-Ela & Sadek [4], Sadek [8] and Tunç [13]. This paper also gives an n-dimensional extension for Abou-El-Ela & Sadek [3] and Chukwu [5]. Furthermore, the present work is the first attempt to obtain sufficient conditions for boundedness of solutions of fifth order vector differential equations.

We need the following notation and definitions :

1. $\lambda_i(A)$, $(i=1,2,...,n)$ are the eigenvalues of the $n x n$ -matrix A.

2. $<X,Y>$ corresponding to any pair X, Y of vectors in R^n is the usual scalar product $\sum_{i=1}^{n} x_i y_i$.

$<X,X> = \|X\|^2$ for arbitrary X in R^n .

3. The Jacobian matrices $J(\Phi(Z,W)\mid Z)$, $J(\Phi(Z,W)\mid W)$, $J(G(Y,Z)\mid Y)$, $J(G(Y,Z)\mid Z)$, $J_H(Y)$ and $J_\Psi(X)$ are given by

$$J(\Phi(Z,W)\mid Z)=\left(\frac{\partial\phi_i}{\partial z_j}\right),$$

$$J(\Phi(Z,W)\mid W)=\left(\frac{\partial\phi_i}{\partial w_j}\right),$$

$$J(G(Y,Z)\mid Y)=\left(\frac{\partial g_i}{\partial y_j}\right),$$

$$J(G(Y,Z)\mid Z)=\left(\frac{\partial g_i}{\partial z_j}\right),$$

$$J_H(Y)=\left(\frac{\partial h_i}{\partial y_j}\right),\quad J_\Psi(X)=\left(\frac{\partial\psi_i}{\partial x_j}\right),\quad (i,j=1,2,...,n).$$

Moreover, let the Jacobian matrices $J(\Phi(Z,W)\mid Z)$, $J(\Phi(Z,W)\mid W)$, $J(G(Y,Z)\mid Y)$, $J(G(Y,Z)\mid Z)$, $J_H(Y)$ and $J_\Psi(X)$ exist and are continuous.

In the case $P\equiv 0$, we shall prove ;

Theorem 1 Further to the basic assumptions on F, Φ, G, H and Ψ, we assume the existence of arbitrary positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and of sufficiently small positive constants $\varepsilon_0, \varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ such that

(i) $\alpha_1 \alpha_2 - \alpha_3 > 0$. $(\alpha_1 \alpha_2 - \alpha_3)\alpha_3 - (\alpha_1 \alpha_4 - \alpha_5)\alpha_1 > 0$,

$$\delta_0 = (\alpha_3 \alpha_4 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 \alpha_4 - \alpha_5)^2 > 0, \tag{1.4}$$

$$\Delta_1 = \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)} - \{\alpha_1 \| J_H(Y) \| - \alpha_5\} > 2\varepsilon\alpha_1 \quad \text{for all } Y \in R^n, \tag{1.5}$$

$$\Delta_2 = \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4 - \alpha_5)} - \frac{(\alpha_1\alpha_4 - \alpha_5)\Gamma(Y)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1} > 0 \quad \text{for all } Y \in R^n, \tag{1.6}$$

where

$$\Gamma(Y) = \int_0^1 J_H(\sigma Y)\,d\sigma. \tag{1.7}$$

(ii) $F(X,Y,Z,W)$ is symmetric and

$$\varepsilon_0 \le \lambda_i\,[\,F(X,Y,Z,W) - \alpha_1 I\,] \le \varepsilon_1 \quad \text{for all } X, Y, Z, W \in R^n \quad (i=1,2\ldots,n).$$

(iii) $\Phi(Z,0)=0$, $J(\Phi(Z,W)\,|\,Z)$ is negative-definite, $J(\Phi(Z,W)\,|\,W)$ is symmetric and

$$0 \le \lambda_i \left(\int_0^1 \{ J(\Phi(Z,\sigma W)|\sigma W) - \alpha_2 I \} d\sigma \right) \le \varepsilon_2 \quad \text{for all } Z, W \in R^n \quad (i=1,2,\ldots,n).$$

(iv) $G(Y,0)=0$, $J(G(Y,Z)\,|\,Z)$ is symmetric, $J(G(Y,Z)\,|\,Y)$ is negative definite and

$$0 \le \lambda_i \left(\int_0^1 \{ J(G(Y,\sigma Z)|\sigma Z) - \alpha_3 I \} d\sigma \right) \le \varepsilon_3 \quad \text{for all } Y, Z \in R^n \quad (i=1,2,\ldots,n).$$

(v) $H(0)=0$, $J_H(Y)$ is symmetric, $\lambda_i \left(\int_0^1 J_H(\sigma Y) d\sigma \right) \ge \alpha_4$, $\| \alpha_4 I - J_H(Y) \| \le \varepsilon_4$ and

$$\lambda_i \left(J_H(Y) - \int_0^1 J_H(\sigma Y) d\sigma \right) \le \frac{\alpha_5 \delta_0}{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \quad \text{for all } Y \in R^n \quad (i=1,2,\ldots,n).$$

(vi) $\Psi(0)=0$, $J_\Psi(X)$ is symmetric and

$$0 \le \lambda_i\,(\alpha_5 I - J_\Psi(X)) \le \varepsilon_5 \quad \text{for all } X \in R^n \quad (i=1,2,\ldots,n).$$

(vii) $J_\Psi(X)$ commutes with $J_\Psi(X')$ for all $X, X' \in R^n$,

$$\lambda_i \left(\int_0^1 J_\Psi(\sigma X) d\sigma \right) \ge \alpha_5' > 0 \quad \text{for all } X \in R^n \quad (i=1,2,\ldots,n).$$

Then every solution $X(t)$ of (1.1) satisfies

$$\|X(t)\| \to 0,\ \|\dot{X}(t)\| \to 0,\ \|\ddot{X}(t)\| \to 0,\ \|\dddot{X}(t)\| \to 0,\ \|X^{(4)}(t)\| \to 0 \quad \text{as } t \to \infty.$$

For the case $P \neq 0$, we shall prove;

Theorem 2 Let all the conditions of Theorem 1 be satisfied, and in addition , we assume that the vector P satisfies

$$\|P(t,X,Y,Z,W,U)\| \le [\delta_1 + \delta_2(\|Y\|+\|Z\|+\|W\|+\|U\|)]\theta(t), \qquad (1.8)$$

where δ_1, δ_2 are positive constants and $\theta(t)$ is a nonnegative and continuous function of t, and satisfies $\int_0^t \theta(s)ds \le A < \infty$, for all $t \ge 0$, where A is a positive constant. Then there exists a constant $D > 0$ such that any solution $(X(t), Y(t), Z(t), W(t), U(t))$ of (1.2) determined by

$$X(0) = X_0,\ Y(0) = Y_0,\ Z(0) = Z_0,\ W(0) = W_0,\ U(0) = U_0,$$

satisfies for all $t \ge 0$,

$$\|X(t)\| \le D,\ \|Y(t)\| \le D,\ \|Z(t)\| \le D,\ \|W(t)\| \le D,\ \|U(t)\| \le D.$$

The following lemmas will be applied by the estimations of a Lyapunov function and its time derivative.

Lemma 1.1 Let A be a real symmetric $n \times n$-matrix and $\alpha' \ge \lambda_i(A) \ge \alpha > 0$ $(i = 1,2,\ldots,n)$, then

$$\alpha'\|X\|^2 \ge \langle AX,X\rangle \ge \alpha\|X\|^2,$$

$$\alpha'^2\|X\|^2 \ge \langle AX,AX\rangle \ge \alpha^2\|X\|^2.$$

Proof See [8].

Lemma 1.2

(I) $$\frac{d}{dt}\int_0^1 \langle \Phi(Z,\sigma W),W\rangle d\sigma \le \langle \Phi(Z,W),U\rangle.$$

(II) $$\frac{d}{dt}\int_0^1 \langle G(Y,\sigma Z),Z\rangle d\sigma \le \langle G(Y,Z),W\rangle.$$

(III) $$\frac{d}{dt}\int_0^1 \langle H(\sigma Y),Y\rangle d\sigma = \langle H(Y),Z\rangle.$$

(IV) $$\frac{d}{dt}\int_0^1 \langle \Psi(\sigma X),X\rangle d\sigma = \langle \Psi(X),Y\rangle.$$

Proof (I) See [13].

(II) $$\frac{d}{dt}\int_0^1 \langle G(Y,\sigma Z),Z\rangle d\sigma = \int_0^1 \langle G(Y,\sigma Z),W\rangle d\sigma + \int_0^1 \langle J(G(Y,\sigma Z)|Y)Z,Z\rangle d\sigma$$

$$+ \int_0^1 \sigma\langle J(G(Y,\sigma Z)|\sigma Z)W,Z\rangle d\sigma. \qquad (1.9)$$

Since $J(G|Z)$ is symmetric from condition *(iv)* we get

$$\int_0^1 \sigma\langle J(G(Y,\sigma Z)|\sigma Z)W,Z\rangle d\sigma = \int_0^1 \sigma\langle J(G(Y,\sigma Z)|\sigma Z)Z,W\rangle d\sigma = \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle G(Y,\sigma Z),W\rangle d\sigma$$

$$= \langle G(Y,Z),W\rangle - \int_0^1 \langle G(Y,\sigma Z),W\rangle d\sigma. \qquad (1.10)$$

From (1.9) and (1.10) we obtain

$$\frac{d}{dt}\int_0^1 \langle G(Y,\sigma Z),Z\rangle d\sigma = \langle G(Y,Z),W\rangle + \int_0^1 \langle J(G(Y,\sigma Z)|Y)Z,Z\rangle d\sigma \le \langle G(Y,Z),W\rangle,$$

since $J(G|Y)$ is negative-definite from assumption *(iv)*.
(III) and *(IV)* can be proved similarly as in *(II)*.

2. The Lyapunov Function V(X,Y,Z,W,U)

The proof of the theorems depends on a scalar differentiable comparison function $V(X,Y,Z,W,U)$. This function and its total time derivative satisfy fundamental inequalities. We define V as follows

$$2V(X,Y,Z,W,U) = \langle U,U\rangle + 2\alpha_1 \langle U,W\rangle + \frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\langle U,Z\rangle + 2\int_0^1 \langle \Phi(Z,\sigma W),W\rangle d\sigma$$

$$+\left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]\langle W,W\rangle + 2\left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_4} - \delta\right\}\langle W,Z\rangle$$

$$+\,2\langle \Psi(X),W\rangle + 2\alpha_1\int_0^1 \langle G(Y,\sigma Z),Z\rangle d\sigma + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_4 - \alpha_1\delta\right]\langle Z,Z\rangle$$

$$+\,2\alpha_2\,\delta\langle Y,Z\rangle + 2\alpha_1\langle Z,H(Y)\rangle - 2\alpha_5\langle Y,Z\rangle + 2\alpha_1\langle \Psi(X),Z\rangle + (\delta\alpha_3 - \alpha_1\alpha_5)\langle Y,Y\rangle$$

$$+\,\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^1 \langle H(\sigma Y),Y\rangle d\sigma + \frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\langle \Psi(X),Y\rangle + 2\delta\langle Y,U\rangle$$

$$+\,2\delta\alpha_1\langle W,Y\rangle + 2\langle W,H(Y)\rangle + 2\delta\int_0^1 \langle \Psi(\sigma X),X\rangle d\sigma, \tag{2.1}$$

where

$$\delta = \left\{\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_4} + \varepsilon\right\} \tag{2.2}$$

The properties of the function $V(X,Y,Z,W,U)$ are summarized in Lemma 2.1 and Lemma 2.2.

Lemma 2.1 Let all the conditions of Theorem 1 be satisfied. Then the function V satisfies

$$V(X,Y,Z,W,U) = 0 \quad \text{at} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 = 0, \tag{2.3}$$

$$V(X,Y,Z,W,U) > 0 \quad \text{if} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0, \tag{2.4}$$

$$V(X,Y,Z,W,U) \to \infty \quad \text{as} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 \to \infty. \tag{2.5}$$

Proof Clearly $V(0,0,0,0,0) = 0$, since $\Phi(Z,0) = G(Y,0) = H(0) = \Psi(0) = 0$. We now prove (2.4) and (2.5). Since $J_H(Y)$ is a symmetric matrix, then it follows that the matrix Γ defined by (1.7) is symmetric. Further, the eigenvalues of Γ are positive because of *(v)*. Consequently the square root $\Gamma^{\frac{1}{2}}$ exists, and this is again symmetric and nonsingular for all $Y \in R^n$. Therefore the function (2.1) admits the representation

$$2V = \left\|U + \alpha_1 W + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}Z + \delta Y\right\|^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left\|Z + \frac{\alpha_5}{\alpha_4}Y\right\|^2$$

$$+\,\frac{\alpha_4(\alpha_1\alpha_4-\alpha_5)}{\alpha_1\alpha_2-\alpha_3}\left\|\frac{(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)}\Gamma^{-\frac{1}{2}}\Psi(X) + \frac{(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)}\Gamma^{\frac{1}{2}}Y + \frac{\alpha_1}{\alpha_4}\Gamma^{\frac{1}{2}}Z + \frac{1}{\alpha_4}\Gamma^{\frac{1}{2}}W\right\|^2$$

$$+\Delta_2 \left\| W+\alpha_1 Z \right\|^2 + \sum_{i=1}^{5} V_i \qquad \textbf{(2.6)}$$

where

$$V_1 = 2\delta \int_0^1 <\Psi(\sigma X), X> d\sigma - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} \left\| \Gamma^{-\frac{1}{2}} \Psi(X) \right\|^2,$$

$$V_2 = \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} \left\{ 2\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma - \langle H(Y), Y\rangle \right\}$$

$$+ \left\{ \delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - \delta^2 \right\} \left\| Y \right\|^2,$$

$$V_3 = \frac{\varepsilon}{\alpha_1} \left\| W \right\|^2 + 2\int_0^1 <\Phi(Z, \sigma W), W> d\sigma - \alpha_2 \left\| W \right\|^2,$$

$$V_4 = 2\alpha_1 \int_0^1 <G(Y, \sigma Z), Z> d\sigma - \alpha_1 \alpha_2 \left\| Z \right\|^2,$$

$$V_5 = \frac{2\varepsilon(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4-\alpha_5)} <Y, Z>. \qquad \textbf{(2.7)}$$

The functions V_1, V_2 and V_3 can be estimated as in [13], in fact the estimates there show that

$$V_1 \geq \varepsilon\alpha_5' \left\| X \right\|^2, \; V_2 \geq \left\{ \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)} \right\} \left\| Y \right\|^2, \; V_3 \geq \frac{\varepsilon}{\alpha_1} \left\| W \right\|^2. \qquad \textbf{(2.8)}$$

Since

$$\frac{\partial}{\partial\sigma}(G(Y, \sigma Z)) = [J(G(Y, \sigma Z) \mid \sigma Z) Z],$$

then we obtain

$$G(Y, Z) = \int_0^1 [J(G(Y, \sigma Z) \mid \sigma Z) Z] d\sigma.$$

Therefore

$$V_4 = 2\alpha_1 \int_0^1 <G(Y, \sigma Z), Z> d\sigma - \alpha_1\alpha_3 \left\| Z \right\|^2$$

$$= 2\alpha_1 \int_0^1 \int_0^1 <\{J[G(Y, \sigma_1 \sigma_2 Z) \mid \sigma_1 \sigma_2 Z] - \alpha_3 I\}\sigma_2 Z, Z> d\sigma_1 \, d\sigma_2 \geq 0, \qquad \textbf{(2.9)}$$

by (iv).

On gathering the estimates (2.7), (2.8) and (2.9) into (2.6) we conclude that

$$2V \geq \left\| U + \alpha_1 W + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} Z + \delta Y \right\|^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2} \left\| Z + \frac{\alpha_5}{\alpha_4} Y \right\|^2 + \Delta_2 \left\| W + \alpha_1 Z \right\|^2$$

$$+ \varepsilon\alpha_5' \left\| X \right\|^2 + \left\{ \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)} \right\} \left\| Y \right\|^2 + \frac{\varepsilon}{\alpha_1} \left\| W \right\|^2 + \frac{2\varepsilon(\alpha_3\alpha_4-\alpha_2\alpha_5)}{(\alpha_1\alpha_4-\alpha_5)} <Y, Z>.$$

It follows from the first six terms of this inequality that there exist positive constants D_i $(i=1,2,3,4,5)$ such that

$$2V \geq D_1 \|X\|^2 + 2D_2 \|Y\|^2 + 2D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2$$

$$+ \frac{2\varepsilon(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4 - \alpha_5)} \langle Y, Z \rangle. \qquad \textbf{(2.10)}$$

The inequality (2.10) is the same as the inequality in [13]. Thus the estimates for V give that

$$2V \geq D_1 \|X\|^2 + D_2 \|Y\|^2 + D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2$$

as in [13].

***Lemma* 2.2** For every solution $(X(t), Y(t), Z(t), W(t), U(t))$ of (1.2) we have

$$\dot{V} \equiv \frac{d}{dt} V(X(t), Y(t), Z(t), W(t), U(t)) \leq 0 \text{ for all } t \geq 0, \text{ and especially} \qquad \textbf{(2.11)}$$

$$\frac{d}{dt} V(X(t), Y(t), Z(t), W(t), U(t)) < 0 \text{ for } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 \neq 0. \qquad \textbf{(2.12)}$$

Proof From (2.1), (1.2) and Lemma 1.2 we have

$$\dot{V} \leq -\{\langle F(X,Y,Z,W)U,U\rangle - \alpha_1 \langle U,U\rangle\}$$

$$-\left[\alpha_1 \langle \Phi(Z,W),W\rangle - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\langle W,W\rangle\right]$$

$$-\left[\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\langle G(Y,Z),Z\rangle - \{\alpha_2\,\delta\langle Z,Z\rangle + \alpha_1 \langle J_H(Y)Z,Z\rangle - \alpha_5\langle Z,Z\rangle\}\right]$$

$$-\left\{\delta\langle Y,H(Y)\rangle - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\langle J_\Psi(X)Y,Y\rangle\right\} - \alpha_1 \langle F(X,Y,Z,W)U,W\rangle$$

$$+\alpha_1^2\langle U,W\rangle - \{\langle G(Y,Z),U\rangle - \alpha_3 \langle Z,U\rangle\} - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\langle F(X,Y,Z,W)U,Z\rangle$$

$$+\left\{\frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right\}\langle U,Z\rangle - \delta\langle F(X,Y,Z,W)Y,U\rangle + \delta\alpha_1 \langle Y,U\rangle$$

$$-\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\{\langle \Phi(Z,W),Z\rangle - \alpha_2 \langle W,Z\rangle\} - \{\alpha_4 \langle W,Z\rangle - \langle W,J_H(Y)Z\rangle\}$$

$$-\alpha_5\langle Y,W\rangle + \langle J_\Psi(X)Y,W\rangle - \delta\{\langle G(Y,Z),Y\rangle - \alpha_3 \langle Y,Z\rangle\}$$

$$-\delta\{\langle \Phi(Z,W),Y\rangle - \alpha_2 \langle W,Y\rangle\} - \alpha_1\,\alpha_5\langle Y,Z\rangle + \alpha_1 \langle J_\Psi(X)Y,Z\rangle. \qquad \textbf{(2.13)}$$

From *(ii)* and Lemma 1.1 we have

$$\langle F(X,Y,Z,W)U,U\rangle - \alpha_1 \langle U,U\rangle \geq \varepsilon_0 \|U\|^2. \qquad \textbf{(2.14)}$$

It follows (2.2), *(iii)* and Lemma 1.1 that

$$\alpha_1 \langle \Phi(Z,W),W\rangle - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\langle W,W\rangle$$

$$= \alpha_1 \left[\int_0^1 <\{J[\Phi(Z,\sigma W) \mid \sigma W] - \alpha_2 I\} W, W> d\sigma \right]$$

$$+ \left\{ \alpha_1\alpha_2 - \alpha_3 + \delta - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right\} <W,W> \geq \varepsilon \| W \|^2 . \qquad \textbf{(2.15)}$$

$$\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} <G(Y,Z),Z> - \{\alpha_2\ \delta<Z,Z> + \alpha_1 <J_H(Y)Z,Z> - \alpha_5<Z,Z>\}$$

$$= \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \int_0^1 <J(G(Y,\sigma Z) \mid \sigma Z)Z,Z> d\sigma - \{\alpha_2\ \delta<Z,Z> + \alpha_1 <J_H(Y)Z,Z> - \alpha_5<Z,Z>\}$$

$$\geq \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} <Z,Z> - \{\alpha_1 \| J_H(Y) \| - \alpha_5\} <Z,Z> - \varepsilon\alpha_2 <Z,Z> \geq \varepsilon\alpha_2 \| Z \|^2$$

by *(iv)*, (2.2), (1.5) and Lemma 1.1.
Similarly, by using (2.2), *(v)*, *(vi)* and Lemma 1.1 we have

$$\delta<Y,H(Y)> - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} < J_\Psi(X)Y,Y>$$

$$= \delta \int_0^1 <Y, J_H(\sigma Y)Y> d\sigma - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} < J_\Psi(X)Y,Y> \geq \varepsilon\alpha_4 \| Y \|^2 .$$

Thus the first four terms in (2.13) are majorizable by

$$- (\varepsilon\alpha_4 \| Y \|^2 + \varepsilon\alpha_2 \| Z \|^2 + \varepsilon \| W \|^2 + \varepsilon_0 \| U \|^2). \qquad \textbf{(2.16)}$$

Now let $R(X,Y,Z,W,U)$ denote the sum of the remaining terms in (2.13). By the hypotheses of Theorem 1 and Schwarz's inequality we obtain

$$| R(X,Y,Z,W,U) \leq D_7(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(\| Y \|^2 + \| Z \|^2 + \| W \|^2 + \| U \|^2) \qquad \textbf{(2.17)}$$

for some constant $D_7 > 0$.
Combining inequalities (2.14) - (2.17) in (2.13), we obtain

$$\dot{V} \leq - (\varepsilon\alpha_4 \| Y \|^2 + \varepsilon\alpha_2 \| Z \|^2 + \varepsilon \| W \|^2 + \varepsilon_0 \| U \|^2)$$

$$+ D_7(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(\| Y \|^2 + \| Z \|^2 + \| W \|^2 + \| U \|^2)$$

$$\leq - \frac{1}{2} \min\{\varepsilon\alpha_4 , \varepsilon\alpha_2 , \varepsilon, \varepsilon_0\}(\| Y \|^2 + \| Z \|^2 + \| W \|^2 + \| U \|^2)$$

if $D_7(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \leq \frac{1}{2} \min\{\varepsilon\alpha_4 , \varepsilon\alpha_2 , \varepsilon, \varepsilon_0\}$.

Hence , we can conclude that

$\dot{V}(t) \leq 0$ for all $t \geq 0$, and especially

$$\frac{d}{dt} V(X(t),Y(t),Z(t),W(t),U(t)) < 0 \quad \text{for} \quad \| X \|^2 + \| Y \|^2 + \| Z \|^2 + \| W \|^2 + \| U \|^2 > 0$$

which proves the lemma.

3. Proof of Theorem 1

The basic properties of $V(X,Y,Z,W,U)$, which we have proved in Lemma 2.1 and Lemma 2.2, justify that the zero solution of (1.2) is asymptotic stability in the large [7].

The system of equations (1.2) is equivalent to the differential equation (1.1). There follows, thus, the original statement of Theorem 1.

4. Proof of Theorem 2

Consider the function V defined as above. Then under the conditions of Theorem 2 the conclusion of Lemma 2.1 can be obtained, that is,

$$2V \geq D_1 \|X\|^2 + D_2 \|Y\|^2 + D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2 \qquad \textbf{(4.1)}$$

and since $P(t,X,Y,Z,W,U)\neq 0$ the conclusion Lemma 2.2 can be revised as follows

$$\dot{V} \leq \langle U+\alpha_1 W+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}Z+\delta Y,\ P(t,X,Y,Z,W,U)\rangle.$$

Since $P(t,X,Y,Z,W,U)$ satisfies (1.8), by Schwarz's inequality we obtain

$$\dot{V} \leq (\|U\|+\alpha_1 \|W\|+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\|Z\|+\delta\|Y\|)\|P(t,X,Y,Z,W,U)\|$$

$$\leq D_8(\|U\|+\|W\|+\|Z\|+\|Y\|)[\delta_1+\delta_2(\|Y\|+\|Z\|+\|W\|+\|U\|)]\theta(t),$$

where $D_8 = \max\left\{1,\alpha_1,\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5},\delta\right\}$.

Hence we can easily obtain

$$\dot{V} \leq [D_9 + D_{10}(\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2)]\theta(t), \qquad \textbf{(4.2)}$$

where $D_9 = 4\delta_1 D_8$, $D_{10} = \delta_1 D_8 + 4\delta_2 D_8$.

Let $D_{11} = \min\left\{\frac{D_2}{2},\frac{D_3}{2},\frac{D_4}{2},\frac{D_5}{2}\right\}$. Then it follows from (4.1) that

$$V \geq D_{11}(\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2). \qquad \textbf{(4.3)}$$

From (4.2) and (4.3) we have

$$\dot{V} \leq D_9\theta(t) + D_{12}V\theta(t),$$

where $D_{12} = \frac{D_{10}}{D_{11}}$.

Gronwall - Bellman inequality yields

$$V \leq D_{13}\exp\left(\int_0^t D_{12}\theta(s)ds\right),$$

where $D_{13} = V(0) + D_9 A$

The proof of Theorem 2 is complete.

REFERENCES

[1] Abou-El-Eia, A.M.A. and Sadek, A.I., EQUADIFF 7, Prague, August 21-25 (1989).

[2] Abou-El-Ela, A.M.A. and Sadek, A.I., Ann. of Diff. Eqs., 6(2), 131-141 (1990).

[3] Abou-El-Ela, A.M.A. and Sadek, A.I., Proc. of Assiut First Intern. Conf., Part V, 15-25 (1990).

[4] Abou-El-Ela, A.M.A. and Sadek, A.I., Ann. of Diff. Eqs., 8(4), 391-400 (1992).

[5] Chukwu, E.N., SIAM J. Math. Anal. 7(2), 176-194 (1976).

[6] Rama Mohana Rao, M., Ordinary Differential Equations, Affiliated East-West Private Limited, London, (1980).

[7] Reissig , R., Sansone, G., and Conti, R., Nonlinear Differential Equations of Higher Order, Noordhoff International Publishing, (1974).

[8] Sadek, A.I., Ann. of Diff. Eqs., 9 (2), 131-140 (1993).

[9] Tiryaki, A. and Tunç, C., Analysis, 16, 325-334 (1996).

[10] Tunç, C., Ann. of Diff. Eqs., 12 (3), 259-266 (1996).

[11] Tunç, C. and Tiryaki, A., Applied Mathematics and Mechanics, 17 (11), 1039-1049 (1996).

[12] Tunç, C., University of Istanbul, Faculty of Science, The Journal of Mathematics, Vol. 54, to appear.

[13] Tunç, C., University of Marmara, The Journal of Sciences, Vol. 10, to appear.

[14] Yu Yuanhong , Journal of Central China Normal University (Natural Sciences), 24(3), 267-273 (1990), (in Chinese).

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