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D-LINES ON THE SURFACES OF PARALLEL MEAN CURVATURE IN ARBITRARY DIMENSIONAL MANIFOLDS OF CONSTANT CURVATURE

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Abstract In this paper, we define D-lines on the surfaces isometrically immersed in arbitrary dimensional manifolds of constant curvature formally and geometrically after having proved that for all arc-length parametrized curves C in M^2 with the same tangent vector $C^0 \in T(M^2)_P$, $P \in M^2$, the function

$$(k_n^\alpha)^0 + k_g t_g^\alpha, \qquad \left((k_n^\alpha)^0 = \frac{dk_n}{ds}\right),$$

for $\alpha=3,\ldots,2+k$, is a function of direction, where k_n, k_g, t_g, and s are, respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C and the upper index α indicates the component of the associated vector of C in the e_α direction for an orthonormal framing $\{e_\alpha\}_{\alpha=3}^{2+k}$ of the normal bundle. By applying Hoffman [8]' s results for surfaces of parallel mean curvature in manifolds of constant curvature, we obtain the following:

(i) If the immersion is totally umbilic, then every line of M^2 is a D-line in the e_α direction for $\alpha=4,\ldots,2+k$ and the differential equation of D-lines can be expressed in lerms of the partial derivatives of the conformal parameter on M^2 in the mean curvature normal direction.

(ii) If the immersion is pseudo-umbilical, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the mean curvature normal direction.

(iii) Away from umbilic points, either every line of M^2 is a D-line in a normal direction perpendicular to the mean curvature normal direction, or the differential equation of D-lines can be expressed in terms of the partial derivates of the conformal parameter on M^2 in the normal direction mentioned above.

We use Chen [1]'s results to reduce the co-dimension and to obtain the final version of the results above.

We also generalise certain classical results for D-lines on surfaces in E^3 and obtain some new ones.

Keywords: D-lines (Darboux lines), parallel (constant) mean curvature

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1.PRELIMINARIES

Let i: $M^2 \to \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion of a 2-dimensional Riemannian manifold M^2 in a (2+k)-dimensional Riemannian manifold $\overline{M}^{2+k}(c)$ of constant sectional curvature c and let X and Y be two tangent vector fields on M^2; i.e., two members of $\Gamma(TM^2)$, the space of smooth sections of TM^2. If $<,>$ denotes the metric tensor on $T\overline{M}^{2+k}$ than that of TM^2 is given by

$$< i_*X, i_*Y> = g(X,Y) \tag{1.1}$$

For all local formulas and computations we consider i as an imbedding thus identify M^2 with $i(M^2)$ and TM^2 with $i_*(TM^2) \subset T\overline{M}^{2+k}$, deleting reference to i and its induced maps wherever possible. As a result, for $X,Y \in T(M^2)_p$ we write $<X,Y>$ for $g(X,Y)$ which we can do via the identification. We consider $T\overline{M}^{2+k}$ restricted to the base space M^2. Let $[\ ]^T$ denote projection in $T\overline{M}^{2+k}$ onto TM^2. Then the normal bundle NM^2 is the bundle whose fibre at P is

$$N(M^2)_p = \{X \in T(\overline{M}^{2+k})_p : [X]^T = 0\}, \tag{1.2}$$

which is the orthogonal complement (with respect to $<,>$) of $T(M^2)_p$ in $T(\overline{M}^{2+k})_p$. We let $[\ ]^N$ denote projection onto NM^2. Let ∇ and $\overline{\nabla}$ be the Riemannian connections of $M^2$and $\overline{M}^{2+k}$ respectively. ∇ is related to $\overline{\nabla}$ by

$$[\overline{\nabla}_X Y]^T = [\nabla_X Y]. \tag{1.3}$$

Then the second fundamental form B of the immersion is given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{1.4}$$

and is a section of $\Gamma(TM^2 \otimes TM^2, NM^2)$, the tensor bundle over M^2 whose fibre at P is the space of bilinear maps from $T(M^2)_p x T(M^2)_p$ to $N(M^2)_p$.

$$B(X,Y) = [\overline{\nabla}_X Y]^N, \tag{1.5}$$

is a normal vector field on M^2 and is symmetric on X and Y.

Let $N \in \Gamma(NM^2)$, we write

$$\overline{\nabla}_X N = A(N,X) + D_X N, \tag{1.6}$$

where $A(N,X)$ and $D_X N$ denote the tangential and normal components of $\overline{\nabla}_X N$. A is a section of $\Gamma(NM^2 \otimes TM^2, TM^2)$ defined by

$$\langle A(N,X),Y\rangle = -\langle B(X,Y),N\rangle, \quad (1.7)$$

and D is the Riemannian connection on NM^2, induced by the immersion, defined by

$$D_X N = [\overline{\nabla}_X N]^N. \quad (1.8)$$

D is easily seen to be compatible with the metric of NM^2.

A normal vector field N on M^2, is said to be parallel in the normal bundle if $D_X N=0$ for all tangent vector fields X.

The mean curvature vector H is the section of NM^2 defined by

$$H=\frac{1}{2} \text{ trace } B. \quad (1.9)$$

The surface M^2 in $\overline{M}^{2+k}(c)$ is said to be minimal if H=0 identically.

The immersion $M^2 \rightarrow \overline{M}^{2+k}(c)$ has parallel mean curvature vector field if H is parallel in the normal bundle. Sometimes this condition will be stated by saying merely that H is parallel.

If the mean curvature vector H and the second fundamental form B satisfy

$$\langle B(X,Y),H\rangle = \lambda(X,Y), \quad (1.10)$$

for all tangent vector fields X,Y at $P \in M^2$ with the same λ, then M^2 is said to be pseudo-umbilical at P. If M^2 is pseudo-umbilical at every point of M^2, then M^2 is called a pseudo-umbilical surface of $\overline{M}^{2+k}(c)$. Similarly, M^2 is totally umbilic at $P \in M^2$ if the second fundamental form is a constant times I, the identity matrix, in every normal direction. If M^2 is totally umbilic at every point of M^2, then M^2 is called a totally umbilic surface of $\overline{M}^{2+k}(c)$.

The curvatures associated with ∇, $\overline{\nabla}$ and D are denoted R, $\overline{R}$ and $\tilde{R}$ respectively. For example, $\tilde{R}$ is given by

$$\tilde{R}(X,Y)N = D_X D_Y N - D_Y D_X N - D_{[X,Y]} N. \quad (1.11)$$

$\tilde{R}$, like R and $\overline{R}$ is skew-symmetric on each fibre on NM^2, bilinear in X and Y. Also, as is obvious, $\tilde{R}(X,Y)_p$ depends only on X_p and Y_p.

A local orthonormal framing of $T\overline{M}^{2+k}$ (resp.,TM^2) we mean 2+k (resp.,two) sections e_i of $T\overline{M}^{2+k}$ (resp.,TM^2) defined on an open set $\overline{U}$(resp.,U) such that $\langle e_i,e_j\rangle=\delta_{ij}$. For an immersed manifold $M^2 \to \overline{M}^{2+k}(c)$, we often consider the framing as defined on $\overline{U}\,|_{M^2}$. It will be convenient to choose framing of $T\overline{M}^{2+k}$ that have the property that $\{e_1,e_2\}$ are sections of $TM^2 \subset T\overline{M}^{2+k}$, and $\{e_3,\ldots,e_{2+k}\}$ are sections of NM^2. Such a framing is called an adapted orthonormal framing. Given a basis of coordinate vectors $\left\{\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2}\right\}$ of TM^2, a completion to a basis of $T\overline{M}^{2+k}$ is a choice of k orthonormal sections $\{e_\alpha\}_{\alpha=3}^{2+k}$ of NM^2. We will call $\left\{\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2},e_3,\ldots,e_{2+k}\right\}$ an adapted coordinate framing of $T\overline{M}^{2+k}$.

For a unit normal section e_α of NM^2 and a framing $\{e_i\}_{i=1}^2$ of TM^2

$$\lambda_{ij}^{\alpha}=\langle B(e_i,e_j),e_\alpha\rangle, \tag{1.12}$$

is the second fundamental form matrix, in the e_α direction, expressed in terms of framing $\{e_i\}_{i=1}^2$ of TM^2. Similarly, for a coordinate basis $\left\{\frac{\partial}{\partial u_i}\right\}_{i=1}^2$ of TM^2,

$$L_{ij}^{\alpha}=\left\langle B\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right),e_\alpha\right\rangle. \tag{1.13}$$

A normal (or adapted) framing for an immersion $M^2 \to \overline{M}^{2+k}(c)$ is said to be an Otsuki frame if $e_3=\frac{H}{\|H\|}$, where H is the mean curvature vector of the immersion.

If there exist an orthonormal framing $\{e_\alpha\}_{\alpha=3}^{2+k}$ of NM^2 such that each e_α is parallel, then we say that the normal bundle is parallel. Such a framing of the normal bundle is called a parallel framing.

2.CONFORMAL IMMERSIONS

An immersion $M^2 \to \overline{M}^{2+k}(c)$, $c \geq 0$, is conformai if there exist coordinates (u_1,u_2) on M^2 such that

$$<\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}>=E(u_1,u_2)\delta_{ij}. \qquad (2.1)$$

On every surface there exist conformal coordinates locally.

Now, let $M^2 \xrightarrow{i} \overline{M}^{2+k}(c)$, $c \geq 0$, be a conformal immersion. If (u_1,u_2) are the conformal coordinates and $ds^2 = E(du_1^2 + du_2^2)$, then

(i) $g_{ij}=E\delta_{ij}$,

and (2.2)

$$g^{ij}=\frac{\delta_{ij}}{E},$$

where

$$(g^{ij})=(g_{ij})^{-1}.$$

(ii) $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = -\Gamma_{22}^1 = \frac{E_{,1}}{2E}$,

and (2.3)

$$\Gamma_{22}^2 = \Gamma_{21}^1 = \Gamma_{12}^1 = -\Gamma_{11}^2 = \frac{E_{,2}}{2E},$$

where

$$E_{,j}=\frac{\partial E}{\partial u_j},$$

and Γ_{ij}^k are the Christoffel symbols given by

$$\nabla_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j} = \sum_{k=1}^{2}\Gamma_{ij}^k\frac{\partial}{\partial u_k}.$$

(iii) The natural orthonormal framing of TM^2 associated with the conformal coordinates (u_1,u_2) is

$$\{e_1,e_2\}=\left\{\frac{\partial}{\partial u_1}\Big/\sqrt{E},\frac{\partial}{\partial u_2}\Big/\sqrt{E}\right\}. \qquad (2.4)$$

(iv)If e_α is a unit normal vector field, then

$$L^{\alpha}_{ij} = E\lambda^{\alpha}_{ij}. \tag{2.5}$$

(v) If $\{e_\alpha\}_{\alpha=3}^{2+k}$ is an orthonormal framing of NM^2, then

$$H = \frac{1}{2}\sum_{\alpha=3}^{2+k}(\lambda^{\alpha}_{11} + \lambda^{\alpha}_{22})e_\alpha$$

$$= \frac{1}{2}\sum_{\alpha=3}^{2}(\frac{L^{\alpha}_{11} + L^{\alpha}_{22}}{E})e_\alpha. \tag{2.6}$$

(vi)If $H \neq 0$, we may choose an Otsuki frame in which $e_3 = \frac{H}{||H||}$. Then

$$H = \frac{1}{2}(\lambda^{3}_{11} + \lambda^{3}_{22})e_3$$

$$= \frac{1}{2}(\frac{L^{3}_{11} + L^{3}_{22}}{E})e_3, \tag{2.7}$$

and

$$\lambda^{\alpha}_{11} + \lambda^{\alpha}_{22} = \frac{L^{\alpha}_{11} + L^{\alpha}_{22}}{E} = 0, \tag{2.8}$$

for $\alpha > 3$.

(vii) If e_α is a unit section of NM^2 which is parallel, then

$$(L^{\alpha}_{11})_{,2} - (L^{\alpha}_{12})_{,1} = \frac{E_{,2}}{2E}(L^{\alpha}_{11} + L^{\alpha}_{22}) = \frac{E_{,2}}{2}(\lambda^{\alpha}_{11} + \lambda^{\alpha}_{22}),$$

and (2.9)

$$(L^{\alpha}_{12})_{,2} - (L^{\alpha}_{22})_{,1} = -\frac{E_{,1}}{2E}(L^{\alpha}_{11} + L^{\alpha}_{22}) = -\frac{E_{,1}}{2}(\lambda^{\alpha}_{11} + \lambda^{\alpha}_{22}).$$

(viii) If e_α is a unit section of NM^2 which is parallel, and if the second fundamental form in the e_α direction satisfies $\frac{L^{\alpha}_{11} + L^{\alpha}_{22}}{E}$=constant, then

$$\varphi_\alpha = \frac{L^{\alpha}_{11} - L^{\alpha}_{22}}{2} - iL^{\alpha}_{12}, \tag{2.10}$$

is an analytic function of $z = u_1 + iu_2$.

(ix) If e_β is any unit vector field with $\langle e_\beta, e_\alpha \rangle = 0$ and e_α is as above, then

$$\frac{\varphi_\beta}{\varphi_\alpha} = f, \tag{2.11}$$

where f is a real function with possible isolated poles. If, in addition, e_β is parallel and $tr\left(L_{ij}^{\beta}\right)/E$ =constant, then

$$\frac{\varphi_\beta}{\varphi_\alpha} = \kappa, \tag{2.12}$$

where κ is a real constant.

(x) If $H \neq 0$ is parallel and $\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, H/\|H\|, e_4, \ldots, e_{2+k}\right\}$ is a coordinate adapted Otsuki frame, then

$$\frac{L_{11}^3 + L_{22}^3}{E} = 2\|H\| = \text{constant},$$

and (2.13)

$$\varphi_3 = \frac{L_{11}^3 - L_{22}^3}{2} - iL_{12}^3,$$

is analytic.

If, in addition, e_α is a unit normal vector field with $\langle e_\alpha, H\rangle = 0$, then

$$\frac{L_{11}^\alpha + L_{22}^\alpha}{E} = 0, \tag{2.14}$$

and φ_α is analytic if e_α is parallel.

A more detailed discussion of the above work can be found in [8].

3. D-LINES

Darboux first studied the problem of determining the lines of a surface whose osculating sphere is tangent to the surface at each point, and which are therefore called D-lines.

Let S be a real surface and let C be a line drawn on S. From the above definition one can see a line C on S will be a D-line if and only if the relation

$$\wp = k_n^{\circ} + k_g t_g = 0, \qquad (k_n^{\circ} = \frac{dk_n}{ds}), \tag{3.1}$$

holds along C, where k_n, k_g, t_g and s are respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C.

Now, let a D-line on a surface S be given in function of any parameter t by

$$u=u(t), \quad v=v(t)$$

u and v being parameters on S. Then the differential equation (3.1) can be expressed in function of the coefficients of the two fundamental forms of S and their partial derivates, namely

$$A_1 u'^3 + A_2 u'^2 v' + A_3 u' v'^2 + A_4 v'^3 + Lu'u'' + M(u'v'' + v'u'') + Nv'v'' -$$

$$(Lu'^2 + 2Mu'v' + Nv'^2)\left[\ln\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right]' = 0 \, , \tag{3.2}$$

where

$$3A_1 = L_u + \frac{L(GE_u - 2FF_u + FE_v) + M(2EF_u - EE_v - FE_u)}{2H^2},$$

$$A_2 = L_v + \frac{M(GE_u - 2FF_u + FE_v) + N(2EF_u - EE_v - FE_u)}{2H^2}$$

$$= M_u + \frac{L(GE_v - FG_u) + M(EG_u - FE_v)}{2H^2},$$

$$A_3 = M_v + \frac{M(GE_v - FG_u) + N(EG_u - FG_v)}{2H^2}$$

$$= N_u + \frac{M(2GF_v - GG_u - FG_v) + N(EG_v - 2FF_v + FG_u)}{2H^2},$$

$$3A_4 = N_v + \frac{M(2GF_v - GG_u - FG_v) + N(EG_v - 2FF_v + FG_u)}{2H^2},$$

and

$$H^2 = EG - F^2, \tag{3.3}$$

where E,F,G are the coefficients of the first fundamental form and L,M,N are the coefficients of the second fundamental form of S.

This introduction is taken from [9].

Let $M^2 \rightarrow \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion given locally in conformai coordinates (u_1, u_2) with conformal parameter E. If $\{e_\alpha\}_{\alpha=3}^{2+k}$ is an orthonormal framing of NM^2 we write the corresponding differential equation in the e_α direction for $\alpha=3,\ldots,2+k$ as

$$\wp^\alpha = A_1 u_1'^3 + A_2 u_1'^2 u_2' + A_3 u_1' u_2'^2 + A_4 u_2'^3 + L_{11}^\alpha u_1' u_1'' + L_{12}^\alpha (u_1' u_2'' + u_2' u_1'') + L_{22}^\alpha u_2' u_2'' -$$

$$(L_{11}^\alpha u_1'^2 + 2L_{12}^\alpha u_1' u_2' + L_{22}^\alpha u_2'^2)\left[\ln\sqrt{Eu_1'^2 + 2Fu_1'u_2' + Gu_2'^2}\right]' = 0, \qquad (3.4)$$

where

$$3A_1 = (L_{11}^\alpha)_{,1} + \frac{L_{11}^\alpha (GE_{,1} - 2FF_{,1} + FE_{,2}) + L_{12}^\alpha (2EF_{,1} - EE_{,2} - FE_{,1})}{2(EG - F^2)},$$

$$A_2 = (L_{12}^\alpha)_{,2} + \frac{L_{12}^\alpha (GE_{,1} - 2FF_{,1} + FE_{,2}) + L_{22}^\alpha (2EF_{,1} - EE_{,2} - FE_{,1})}{2(EG - F^2)}$$

$$= (L_{12}^\alpha)_{,1} + \frac{L_{11}^\alpha (GE_{,2} - FG_{,1}) + L_{12}^\alpha (EG_{,1} - FE_{,2})}{2(EG - F^2)},$$

$$A_3 = (L_{12}^\alpha)_{,2} + \frac{L_{12}^\alpha (GE_{,2} - FG_{,1}) + L_{22}^\alpha (EG_{,1} - FE_{,2})}{2(EG - F^2)}$$

$$= (L_{22}^\alpha)_{,1} + \frac{L_{11}^\alpha (2GF_{,2} - 2GG_{,1} + FG_{,2}) + L_{12}^\alpha (EG_{,2} - 2FF_{,2} + FG_{,1})}{2(EG - F^2)},$$

$$3A_4 = (L_{22}^\alpha)_{,2} + \frac{L_{12}^\alpha (2GF_{,2} - 2GG_{,1} - FG_{,2}) + L_{22}^\alpha (EG_{,2} - 2FF_{,2} + FG_{,1})}{2(EG - F^2)}. \qquad (3.5)$$

We first make our formal definition.

DEFINITION: Let $M^2 \rightarrow \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion and let C be a line drawn on M^2. C is said to be a D-line in the e_α direction if and only if the differential equation (3.4) holds along C, where $\{e_\alpha\}_{\alpha=3}^{2+k}$ is an orthonormal framing of NM^2.

Let $M^2 \rightarrow \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion given locally in conformal coordinates (u_1, u_2) with conformal parameter E. If $\{e_\alpha\}_{\alpha=3}^{2+k}$ is an orthonormal framing of NM^2 we write (3.4) and (3.5) as

$$\wp^\alpha = A_1 u_1'^3 + A_2 u_1'^2 u_2' + A_3 u_1' u_1'^2 + A_4 u_2'^3 + L_{11}^\alpha u_1' u_1'' + L_{12}^\alpha (u_1' u_2'' + u_2' u_1'') + L_{22}^\alpha u_2' u_2'' -$$

$$(L_{11}^\alpha u_1'^2 + 2L_{12}^\alpha u_1' u_2' + L_{22}^\alpha u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0, \qquad (3.6)$$

where

$$3A_1=(L^\alpha_{11})_{,1}+\frac{L^\alpha_{11}E_{,1}-L^\alpha_{12}E_{,2}}{2E},$$

$$A_2=(L^\alpha_{11})_{,2}+\frac{L^\alpha_{12}E_{,1}-L^\alpha_{22}E_{,2}}{2E}$$

$$=(L^\alpha_{12})_{,1}+\frac{L^\alpha_{11}E_{,2}+L^\alpha_{12}E_{,1}}{2E},$$

$$A_3=(L^\alpha_{12})_{,2}+\frac{L^\alpha_{12}E_{,2}+L^\alpha_{22}E_{,1}}{2E}$$

$$=(L^\alpha_{22})_{,1}+\frac{L^\alpha_{12}E_{,2}-L^\alpha_{11}E_{,1}}{2E},$$

$$3A_4=(L^\alpha_{22})_{,2}+\frac{L^\alpha_{22}E_{,2}-L^\alpha_{12}E_{,1}}{2E}, \qquad (3.7)$$

for $\alpha=3,\ldots,2+k$.

A point where φ_α is real is a point where the second fundamental form in the e_α direction is diagonaiized. For such an α, by substituting $L^\alpha_{12}=0$ in (3.6) and (3.7) we obtain

$$\wp^\alpha=A_1 u_1'^3+A_2u_1'^2u_2'+A_3u_1'u_2'^2+A_4u_2'^3+L^\alpha_{11}u_1'u_1''+L^\alpha_{22}u_2'u_2''-$$

$$(L^\alpha_{11}u_1'^2+L^\alpha_{22}u_2'^2)\left[\ln\sqrt{E(u_1'^2+u_2'^2)}\right]'=0, \qquad (3.8)$$

where

$$3A_1=(L^\alpha_{11})_{,1}+L^\alpha_{11}\frac{E_{,1}}{2E},$$

$$A_2=(L^\alpha_{11})_{,2}-L^\alpha_{22}\frac{E_{,2}}{2E}$$

$$=L^\alpha_{11}\frac{E_{,2}}{2E},$$

$$A_3=L^\alpha_{22}\frac{E_{,1}}{2E}$$

$$=(L^\alpha_{22})_{,1}-L^\alpha_{11}\frac{E_{,1}}{2E},$$

$$3A_4=(L^\alpha_{22})_{,2}+L^\alpha_{22}\frac{E_{,2}}{2E}. \qquad (3.9)$$

If e_α is a unit normal vector field which is parallel in NM^2, then the equations (2.9) becomes

$$(L^\alpha_{11})_{,2} = \frac{E_{,2}}{2E}(L^\alpha_{11} + L^\alpha_{22}),$$

and (3.10)

$$(L^\alpha_{22})_{,1} = \frac{E_{,1}}{2E}(L^\alpha_{11} + L^\alpha_{22}),$$

for the mentioned α.

If, in addition, the second fundamental form in the e_α direction satisfies $\frac{L^\alpha_{11} + L^\alpha_{22}}{E} = \text{constant}$,

then φ_α satisfies the Cauchy-Riemann equations. Thus

$$\left(\frac{L^\alpha_{11} - L^\alpha_{22}}{2}\right)_{,1} = -(L^\alpha_{12})_{,2} = 0,$$

and (3.11)

$$\left(\frac{L^\alpha_{11} - L^\alpha_{22}}{2}\right)_{,2} = (L^\alpha_{12})_{,1} = 0,$$

and then

$$(L^\alpha_{11})_{,1} = (L^\alpha_{22})_{,1},$$
$$(L^\alpha_{11})_{,2} = (L^\alpha_{22})_{,2}. \qquad (3.12)$$

Hence we rewrite the equations (3.9) as

$$3A_1 = \frac{E_{,1}}{2E}(2L^\alpha_{11} + L^\alpha_{22}),$$

$$A_2 = \frac{E_{,2}}{2E}L^\alpha_{11},$$

$$A_3 = \frac{E_{,1}}{2E}L^\alpha_{22},$$

$$3A_4 = \frac{E_{,2}}{2E}(L^\alpha_{11} + 2L^\alpha_{22}), \qquad (3.13)$$

and the equation (3.8) becomes

$$\wp^\alpha = \frac{1}{3}\frac{E_{,1}}{2E}(2L^\alpha_{11} + L^\alpha_{22})u_1'^3 + \frac{E_{,2}}{2E}L^\alpha_{11}u_1'^2u_2' + \frac{E_{,1}}{2E}L^\alpha_{22}u_1'u_2'^2 + \frac{1}{3}\frac{E_{,2}}{2E}(L^\alpha_{11} + 2L^\alpha_{22})u_2'^3 +$$

$$L^\alpha_{11}u_1'u_1'' + L^\alpha_{22}u_2'u_2'' - (L^\alpha_{11}u_1'^2 + L^\alpha_{22}u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0. \qquad (3.14)$$

To give the geometric interpretation of the above work we introduce the notion of covariant differentiation in a tensor bundle of multilinear maps.

DEFINITION: Let T_i, $i=1,\ldots,r+1$, be tensor bundles over M^2 with Riemannian metrics g_i and associated connections ∇^i (e.g.,(TM^2, g,∇)). If S is a section of $H(\bigotimes_{i=1}^{r} T_i,T_{r+1})$ and X is a section of TM^2, then the covariant derivative of S along X, denoted $\nabla_X S$, is the section of $H(\bigotimes_{i=1}^{r} T_i,T_{r+1})$ defined by

$$(\nabla_X S)(Y_1,\ldots,Y_{r+1})=\nabla_X^{r+1}(S(Y_1,\ldots,Y_r))-\sum_{i=1}^{r} S(Y_1,\ldots,Y_{i-1},\nabla_X^i Y_i,Y_{i+1},\ldots,Y_r), \quad (3.15)$$

where Y_i is a section of T_i.

EXAMPLE : B, the second fundamental form of the immersion $M^2\to\overline{M}^{2+k}(c)$. B is a section of $H(TM^2\otimes TM^2,NM^2)$ defined by

$$(\nabla_X B)(Y,Z)=D_X B(Y,Z)-B(\nabla_X Y,Z)-B(Y,\nabla_X Z). \quad (3.16)$$

This immediately yields:

PROPOSITION : Let $M^2\to\overline{M}^{2+k}(c)$, $c\geq 0$, be an isometric immersion and let $\{e_\alpha\}_{\alpha=3}^{2+k}$ be an orthonormal framing of the normal bundle. For all arc-length parametrized curves C in M^2 with the same tangent vector $C^\circ\in T(M^2)_p, P\in M^2$, the quantity

$$(k_n^\alpha)^\circ+k_g t_g^\alpha, \qquad ((k_n^\alpha)^\circ=\frac{dk_n^\alpha}{ds})), \quad (3.17)$$

for $\alpha=3,\ldots,2+k$, depends only on C°, where k_n, k_g, t_g, and s are respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C and the upper index α indicates the component of the associated vector of C in the e_α direction.

Proof: Let X be a unit vector field on M^2 which extends C°, and let Y be the perpendicular unit vector field with {X,Y} positively oriented. The equations

$$k_n(X)=B(X,X),$$
$$t_g(X)=B(X,Y), \quad (3.18)$$

and the fact

$$[C^{\circ\circ}(s)]^T=[\overline{\nabla}_{\frac{dC}{ds}}C^\circ(s)]^T=\nabla_{\frac{dC}{ds}}C^\circ, \quad (3.19)$$

that gives

$$\nabla_X X = k_g Y, \qquad (3.20)$$

along C, leads us to the following. From the equality

$$<(\nabla_X B)(X,X),e_\alpha> = <D_X B(X,X),e_\alpha> - 2<B(\nabla_X X,X),e_\alpha> \qquad (3.21)$$

$$= X< B(X,X), e_\alpha> - < B(X,X), D_X e_\alpha> = 2<B(\nabla_X X, X), e_\alpha>,$$

where $\{e_\alpha\}_{\alpha=3}^{2+k}$ is an orthonormal framing of NM^2, we obtain that

$$<(\nabla_X B)(X,X),e_\alpha> + <B(X,X), D_X e_\alpha> = X< B(X,X), e_\alpha> - 2<B(\nabla_X X,X),e_\alpha>$$

$$= X(k_n^\alpha) - 2<B(k_g Y,X),e_\alpha>$$

$$= (k_n^\alpha)^\circ - 2k_g t_g^\alpha. \qquad (3.22)$$

Hence

$$<(\nabla_X B)(X,X),e_\alpha> + <B(X,X), D_X e_\alpha> + 3 <B(\nabla_X X,X), e_\alpha> = (k_n^\alpha)^\circ + 2k_g t_g^\alpha, \qquad (3.23)$$

which shows that the expression (3.17) depends only on $X=C^\circ$.

We thus show that, like k_n and t_g, the quantity in (3.17)

$$(k_n^\alpha)^\circ + k_g t_g^\alpha$$

also depends only on the direction. Note that if e_α is parallel, then

$$<(\nabla_X B)(X,X),e_\alpha> + 3 <B(\nabla_X X,X), e_\alpha> = (k_n^\alpha)^\circ + k_g t_g^\alpha. \qquad (3.24)$$

What we have done is to observe that: Just as k_n and t_g can be expressed in terms of the tensor B, the expression (3.17) can be expressed in terms of the covariant derivatives of B. It is now a straightforward calculation to see that the entire left hand side of (3.17) is (3.4).

We now have the necessary groundwork done to make our geometric definition.

DEFINITION: Let $M^2 \to \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion and let C be a line drawn on M^2. C is said to be a D-line in the e_α direction if and only if the relation

$$\wp^\alpha = (k_n^\alpha)^\circ + k_g t_g^\alpha = 0, \qquad \left((k_n^\alpha)^\circ = \frac{dk_n^\alpha}{ds}\right), \qquad (3.25)$$

holds along C, where k_n, k_g, t_g, and s are, respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C and the upper index α indicates the component of the associated vector of C in the e_α direction for an orthonormal framing $\{e_\alpha\}_{\alpha=3}^{2+k}$ of NM^2.

We obtain the following :

THEOREM : Let $M^2 \to \overline{M}^{2+k}(c)$, $c \geq 0$, be an isometric immersion (given in conformal coordinates (u_1,u_2) with conformal parameter E) with non-zero, parallel mean curvature. Then,

(i) If $\varphi_3=\varphi_4=\ldots=\varphi_{2+k}\equiv 0$, then every line of M^2 is a D-line in the e_α direction for $\alpha=4,\ldots,2+k$ and the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_3 direction.

(ii) If $\varphi_3\equiv 0$ and $\varphi_\alpha \not\equiv 0$ for an α such that $4\leq\alpha\leq 2+k$ and e_α is parallel in the normal bundle, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_3 and e_α direction.

(iii) If $\varphi_3 \not\equiv 0$ and $\varphi_\alpha\equiv 0$ for an α such that $4\leq\alpha\leq 2+k$ and e_α is parallel in the normal bundle, then every line of M^2 is a D-line in the e_α direction.

(iv) If $\varphi_3 \not\equiv 0$ and $\varphi_\alpha \not\equiv 0$ for an α such that $4\leq\alpha\leq 2+k$ and e_α is parallel in the normal bundle, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_α direction.

Proof. (i) If $\varphi_3=\varphi_4=\ldots=\varphi_{2+k}\equiv 0$, then

$$(L^3_{ij}) = \begin{bmatrix} L^3_{11} & 0 \\ 0 & L^3_{11} \end{bmatrix} \quad \text{and} \quad (L^\alpha_{ij}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.26}$$

Hence $\wp^\alpha \equiv 0$ by (3.8) for $\alpha=4,\ldots,2+k$ and (3.14) reduces to

$$\wp^3 = \frac{E_{,1}}{2E} L^3_{11} u_1'^3 + \frac{E_{,2}}{2E} L^3_{11} u_1'^2 u_2' + \frac{E_{,1}}{2E} L^3_{11} u_1' u_2'^2 + \frac{E_{,2}}{2E} L^3_{11} u_2'^3 + L^3_{11} u_1' u_1'' + L^3_{11} u_2' u_2'' -$$

$$L^3_{11}(u_1'^2 + u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0, \tag{3.27}$$

for $\alpha=3$. Since $L^3_{11} \neq 0$, we have

$$\wp^3 = \frac{E_{,1}}{2E} u_1'^3 + \frac{E_{,2}}{2E} u_1'^2 u_2' + \frac{E_{,1}}{2E} u_1' u_2'^2 + \frac{E_{,2}}{2E} u_2'^3 + u_1' u_1'' + u_2' u_2'' -$$

$$(u_1'^2 + u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0, \tag{3.28}$$

Thus, every line of M^2 is a D-line in the e_α direction for $\alpha=4,\ldots,2+k$ and the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_3 direction.

(ii) If $\varphi_3\equiv 0$ and $\varphi_\alpha \not\equiv 0$ for an α such that $4\leq\alpha\leq 2+k$, then

$$(L^3_{ij}) = \begin{bmatrix} L^3_{11} & 0 \\ 0 & L^3_{11} \end{bmatrix} \quad \text{and} \quad (L^\alpha_{ij}) = \begin{bmatrix} L^\alpha_{11} & 0 \\ 0 & -L^\alpha_{11} \end{bmatrix}. \qquad (3.29)$$

Hence (3.14) reduces to (3.28) for α=3, and reduces to

$$\wp^\alpha = \frac{1}{3}\frac{E_{,1}}{2E} L^\alpha_{11} u_1'^3 + \frac{E_{,2}}{2E} L^\alpha_{11} u_1'^2 u_2' - \frac{E_{,1}}{2E} L^\alpha_{11} u_1' u_2'^2 - \frac{1}{3}\frac{E_{,2}}{2E} L^\alpha_{11} u_2'^3 + L^\alpha_{11} u_1' u_1'' - L^\alpha_{11} u_2' u_2'' -$$

$$L^\alpha_{11}(u_1'^2 + u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0, \qquad (3.30)$$

for the α mentioned above for which e_α is parallel in the normal bundle. Since $L^3_{11} \neq 0$, we have

$$\wp^\alpha = \frac{1}{3}\frac{E_{,1}}{2E} u_1'^3 + \frac{E_{,2}}{2E} u_1'^2 u_2' - \frac{E_{,1}}{2E} u_1' u_2'^2 - \frac{1}{3}\frac{E_{,2}}{2E} u_2'^3 + u_1' u_1'' - u_2' u_2'' -$$

$$(u_1'^2 + u_2'^2)\left[\ln\sqrt{E(u_1'^2 + u_2'^2)}\right]' = 0, \qquad (3.31)$$

Thus, the differential equation of D-lines can be expressed in terms of the partial derivates of the conformal parameter on M^2 in the e_3 and e_α direction.

(iii) If $\varphi_3 \not\equiv 0$ and $\varphi_\alpha \equiv 0$ for an α such that $4 \leq \alpha \leq 2+k$, then

$$(L^3_{ij}) = \begin{bmatrix} L^3_{11} & 0 \\ 0 & L^3_{22} \end{bmatrix} \quad \text{and} \quad (L^\alpha_{ij}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad (3.32)$$

Hence, $\wp^\alpha$=0 by (3.8) for the α mentioned above.

Thus, every line of M^2 is a D-line in the e_α direction.

(iv) If $\varphi_3 \not\equiv 0$ and $\varphi_\alpha \not\equiv 0$ for an α such that $4 \leq \alpha \leq 2+k$, then

$$(L^3_{ij}) = \begin{bmatrix} L^3_{11} & 0 \\ 0 & L^3_{22} \end{bmatrix} \quad \text{and} \quad (L^\alpha_{ij}) = \begin{bmatrix} L^\alpha_{11} & 0 \\ 0 & -L^\alpha_{11} \end{bmatrix}. \qquad (3.33)$$

Hence (3.14) reduce to (3.31) for the α mentioned above.

Thus, the differential equation of D-lines can be expressed in terms of the partial derivates of the conformal parameter on M^2 in the e_α direction.

REMARK: Since φ_3 is analytic, either it is identically zero or has only isolated zeros. Thus, either the immersion is pseudo-umbilic or the pseudo-umbilic and totaly umbilic points are isolated. As a corollary of the theorem we have the following:

COROLLARY: Let $M^2 \to \overline{M}^{2+k}(c)$, c≥0, be an isometric immersion (given in conformal coordinates (u_1,u_2) with conformal parameter E) with non-zero, parallel mean curvature. Then,

(i) If the immersion is totaly umbilic, then every line of M^2 is a D-line in the e_α direction for $\alpha=4,\ldots,2+k$ and the differential equation of D-lines can be expressed in terms of the partial derivates of the conformal parameter on M^2 in the e_3 direction.

(ii) If the immersion is pseudo-umbilical, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_3 direction. If, in addition, there exist a parallel unit normal section e_α of NM^2 for which φ_α is real and non-zero, where $4\leq\alpha\leq 2+k$, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_α direction.

(iii) Away from umbilic points, if there exist a parallel unit normal section e_α of NM^2 where $4\leq\alpha\leq 2+k$, then either every line of M^2 is a D-line in the e_α direction (if $\varphi_\alpha\equiv 0$) or the differential equation D-lines can be expressed in terms of the partial derivates of the conformal parameter on M^2 in the e_α direction (if $\varphi_\alpha \not\equiv 0$).

Proof: The proof follows immediately from the theorem.

We also have the following version of the theorem:

THEOREM: Let $M^2 \to \overline{M}^{2+k}(c)$, $c\geq 0$, be an isometric immersion (given in conformal coordinates (u_1,u_2) with conformal parameter E) with non-zero, parallel mean curvature. If M^2 is not a minimal surface of a hypersphere of $\overline{M}^{2+k}(c)$, then either every line of M^2 is a D-line in the e_α direction or the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_α direction for $\alpha \neq 3$, where $\{e_\alpha\}_{\alpha=3}^{2+k}$ is a parallel framing of the normal bundle.

Proof: Since the mean curvature vector H of the immersion is non-zero we may choose an Otsuki frame of the normal bundle bundle: $\{e_3=\frac{H}{\|H\|},e_4,\ldots,e_{2+k}\}$. Then, in terms of the basis of coordinate vectors $\left\{\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2}\right\}$ of TM^2, the equation of D-lines on M^2 is given by (3.16). If M^2 is not a minimal surface of a hypersphere of $\overline{M}^{2+k}(c)$, then the curvature of the normal connection is zero [1] which is precisely the condition for simultaneous diagonalization [8]. Thus the equation (3.6) reduces to (3.8). Since the triviality of the normal connection is equivalent to the parallelity of the normal bundle [2] the equations (3.10) are valid. Also, since each φ_α is analytic by (x), the equations (3.11) and (3.12) are valid too. Therefore we have (3.14). Now, either $\varphi_\alpha \equiv 0$ and every line of M^2 is a D-line in the e_α direction or $\varphi_\alpha \not\equiv 0$ and the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter in the e_α direction for $4 \le \alpha \le 2+k$.

The immersion $M^2 \to \overline{M}^{2+k}(c)$ is said to be totally geodesic at $P \in M^2$ if the second fundamental form is identically zero in every normal direction. If M^2 is totally geodesic at every point of M^2, then M^2 is called a totally geodesic surface of $\overline{M}^{2+k}(c)$. Since $\wp^\alpha \equiv 0$ on a totally geodesic surface for $\alpha=3,4,\ldots,2+k$, we deduce that: All lines of a totally geodesic surface are D-lines in every normal direction.

4. REDUCING THE CO-DIMENSION

On an analytic function $\varphi \not\equiv 0$ of $z=u_1+iu_2$, defined in a neighbourhood of the origin in the (u_1,u_2)-plane, and constants α,β with $\alpha>0$, Hoffman [8] proved that, up to euclidian motions and isothermal coordinates $E(u_1,u_2)$, locally there exist one and only one surface in $\overline{M}^4(c)$, denoted by $M^2(\varphi,\alpha,\beta)$, with parallel mean curvature H such that $\alpha=\|H\|$ and $\varphi=\varphi_3$, $\beta\varphi=\varphi_4$ where φ_3 and φ_4 are given in (viii). These surfaces are, easy to check that, contained in either in an affine 3-space or in a great or small 3-sphere of $\overline{M}^4(c)$ and they are neither minimal surfaces in $\overline{M}^4(c)$ nor minimal surfaces of hyperspheres of $\overline{M}^4(c)$. It is then possible to classify surfaces, isometrically immersed in constant curvature manifolds, with parallel mean curvature vector as following

(i) Minimal surfaces of $\overline{M}^{2+k}(c)$,

(ii) Minimal surfaces of a hypersphere of $\overline{M}^{2+k}(c)$,

(iii) Surfaces in an affine 3-space or in a great or small 3-sphere of $\overline{M}^4(c)$ and locally given by Hoffman surfaces [1], [4].

D-lines on the surfaces of parallel mean curvature in four dimensional manifolds of constant curvature has been studied seperately [7]. For arbitrary co-dimension we now have the final version of the last theorem.

THEOREM: Let $M^2 \to \overline{M}^{2+k}(c)$, $c \ge 0$, be an isometric immersion (given in conformal coordinates (u_1,u_2) with conformal parameter E) with non-zero, parallel mean curvature. If M^2 is not a minimal surface of a hypersphere of $\overline{M}^{2+k}(c)$, then either every line of M^2 is a D-line in the e_4 direction or the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_4 direction.

Proof. Since M^2 is neither a minimal surface of $\overline{M}^{2+k}(c)$ nor a minimal surface of a hypersphere of $\overline{M}^{2+k}(c)$, M^2 is contained in an affine 3-space or in a great or small 3-sphere of $\overline{M}^4(c)$ and locally given by Hoffman surfaces. Let $\{e_3=\frac{H}{\|H\|},e_4\}$ be an Otsuki frame of the normal bundle. Since $H\neq 0$ parallel and co-dimension is two the normal bundle is parallel and φ_3 and φ_4 are both analytic. Now, $\varphi_3\not\equiv 0$ since pseudo-umbilic immersions with non-zero, parallel mean curvature lie minimaly in a hypersphere of $\overline{M}^{2+k}(c)$. Hence, either $\varphi_4\equiv 0$ and every line of M^2 is a D-line in the e_4 direction or $\varphi_4\not\equiv 0$ and the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_4 direction.

We end with the argument for the proposition of section 3 that actually suffice to prove a slightly more general statement:

THEOREM: Let $M^2\rightarrow\overline{M}^{2+k}(c)$, $c\geq 0$, be an isometric immersion and let $\{e_\alpha\}_{\alpha=3}^{2+k}$ be orthonormal frame of the normal bundle. For all arc-length parametrized curves C in M^2 with the same tangent vector $C^{\circ}\in T(M^2)_p, P\in M^2$, the expression

$$\frac{d}{ds}(\|k_n\|^2)+2k_g\sum_{\alpha=3}^{2+k}k_n^\alpha t_g^\alpha\,, \tag{4.1}$$

is a function of direction, where k_n, k_g, t_g and s are, respectively the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of C and the upper index α indicates the component of the associated vector of C in the e_α direction.

Proof: An easy calculation shows that

$$\frac{d}{ds}(\|k_n\|^2)+2k_g\sum_{\alpha=3}^{2+k}k_n^\alpha t_g^\alpha=2\sum_{\alpha=3}^{2+k}k_n^\alpha\wp^\alpha\,. \tag{4.2}$$

In the case of a hypersurface; i.e., if the codimension is one, assuming that $k_n\neq 0$ and dividing (4.2) through by 2 k_n we get (3.1)

5.DISCUSSION

For surfaces in E^3, the condition of constant mean curvature has been well-studied. For hypersurfaces, the requirement that H be parallel is equivalent to H being of constant length. In this paper, we are mainly interested in immersions with codimension is greater than two. There, parallel mean curvature is a stronger condition, it implies $\|H\|$=constant .

The 2-sphere in euclidian (2+k)-space is totally umbilic. Hence, all lines of the 2-sphere are D-lines in the normal direction perpendicular to the mean curvature normal direction. Conversely, a totally umbilic surface M^2 of $\overline{M}^{2+k}(c)$ is a standart sphere of radius $1/\|H\|$ in the euclidian case, and a great or small sphere in the case c>0.

Totally umbilic implies pseudo-umbilic but pseudo-umbilic does not imply totally umbilic, take a flat Clifford torus in E^4 which is an immersion of E^2 into the unit sphere $S^3(1)\subset E^4$, given by

$$X: E^2 \to E^4$$

$$(u_1,u_2) \to \left(\frac{\sqrt{2}}{2}\cos\sqrt{2}\,u_1, \frac{\sqrt{2}}{2}\sin\sqrt{2}\,u_1, \frac{\sqrt{2}}{2}\cos\sqrt{2}\,u_2, \frac{\sqrt{2}}{2}\sin\sqrt{2}\,u_2\right),(5.1)$$

whose image $X(E^2)$ is a torus T^2 with sectional curvature zero in the induced metric. A simple calculation shows that, for an Otsuki framing $\{e_3=\frac{H}{\|H\|}, e_4\}$, this immersion is pseudo-umbilic but not totally umbilic. These various types of umbilicity may be confusing to the reader familiar only with hypersurfaces in euclidian space. There, pseudo-umbilic=umbilic = totally umbilic since there is only one normal direciton.

In the case that $\overline{M}^{2+k}=E^{2+k}$, the linear subspaces and their translates are evidently totally geodesic submanifolds. Hence, all lines of the planes are D-lines in every normal direction. For c>0, i.e., $\overline{M}^{2+k}\approx S^{2+k}(1/\sqrt{c})\subset E^{(2+k)+1}$, the intersections of linear subspaces of $E^{(2+k)+1}$ with $S^{2+k}(1/\sqrt{c})$ are totally geodesic submanifolds. Hence, all lines of the small or great (k+1)-spheres of $S^{2+k}(1/\sqrt{c})$ are D-lines in every normal direction. These includes some of the Hoffman surfaces. These surfaces are, easy to check that, contained in a 3-dimensional totally geodesic subspace of $\overline{M}^4(c)$ if $\beta=0$.

An immersion $M^2 \to E^4$ is said to be a standard product immersion if M^2 is a piece of the standard product immersion of $S^1(r)xS^1(\rho)$ into E^4. ρ may take the value of $+\infty$, so this includes right circular cylinders. If r=ρ, then M^2 is a piece of the Clifford torus. An immersion $M^2 \to$

$S^4(1/\sqrt{c})$ is a standard product immersion if there is a 4-dimensional affine subspace in E^5 such that M^2 lies in it and is a standard product immersion in the euclidian sense. When $|\varphi / E|$ is constant, Hoffman surfaces are pieces of the standard product immersion. Hence, either every line of M^2 is a D-line in the e_4 direction ($\rho=\infty$) or the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the e_3 and e_4 direction ($r=\rho$, $r\neq\rho$).

For the immersions $M^2 \to \overline{M}^{2+k}(c)$ with non-zero, parallel mean curvature and constant Gauss curvature K, it is shown [8] that, if the normal bundle is parallel, then K may take only the values 0 or $\|H\|^2+c$. If $K\equiv\|H\|^2+c$ and $c\geq 0$, then M^2 is immersed as a piece of the standard 2-sphere. An immersion $M^2 \to \overline{M}^{2+k}(c)$, $c\geq 0$ with non-zero, parallel mean curvature and $K\equiv 0$ is a standard product immersion $S^1(r) \times S^1(\rho)$, $0 < r < \infty$, $0 < p \leq \infty$, where $\|H\|^2 = \frac{1}{r^2} + \frac{1}{\rho^2}$ [3].

For the complete surfaces $M^2 \to E^{2+k}$ with non-zero, parallel mean curvature and Gauss curvature K which does not change sign, M^2 is either a product surface of two plane circles or a product surface of a straight line and a plane circle [1]. An immersion $M^2 \to \overline{M}^{2+k}(c)$, $c\geq 0$ with non-zero, parallel mean curvature and constant Gauss curvature K which does not change sign must be a sphere of radius $\frac{1}{(\|H\|^2+c)^{1/2}}$ or a product of circles $S^1(r) \times S^1(\rho)$, $0 < r < \infty$, $0< \rho \leq \infty$, with the standard product immersion [7].

Since we are mainly interested in surfaces with non-zero, parallel mean curvature, in this paper, no result has been stated for the case $H\equiv 0$. Only, using the fact that, an immersion $M^2 \to \overline{M}^{2+k}(c)$, $c\geq 0$ with non-zero, parallel mean curvature is pseudo-umbilical $\Leftrightarrow$ M^2 lies minimally in some hypersphere of $\overline{M}^{2+k}(c)$, we observe that if M^2 is a minimal surface of a hypersphere of $\overline{M}^{2+k}(c)$, then the differential equation of D-lines can be expressed in terms of the partial derivatives of the conformal parameter on M^2 in the mean curvature normal direction.

A closed, oriented surface M^2 of genus zero immersed in $\overline{M}^{2+k}(c)$, $c\geq 0$, with non-zero, parallel mean curvature is pseudo-umbilical and lies minimally in a hypersphere of radius $\frac{1}{(\|H\|^2+c)^{1/2}}$ [8].

Finally, a compact; flat surface in E^{2+k} with non-zero, parallel mean curvature is a product of two plane circles [1].

6. ON CERTAIN CASES OF INTEGRATION

Suppose that

$$u_1(t)\equiv t \tag{6.1}$$

In this case

$$u_1'=1,\ u_1''=0,\ u_2'=\frac{du_2}{du_1},\ u_2''=\frac{d^2u_2}{du_1^{\ 2}}, \tag{6.2}$$

and the equation (3.14) becomes

$$\wp^{\alpha} = \frac{1}{3}\frac{E_{,1}}{2E}(2L^{\alpha}_{11}+L^{\alpha}_{22})+\frac{E_{,2}}{2E}L^{\alpha}_{11}u_2'+\frac{E_{,1}}{2E}L^{\alpha}_{22}u_2'^{2}+\frac{1}{3}\frac{E_{,2}}{2E}(L^{\alpha}_{11}+2L^{\alpha}_{22})u_2'^{3}+L^{\alpha}_{22}u_2'u_2''-$$

$$(L^{\alpha}_{11}+L^{\alpha}_{22}u_2'^{2})[\ln\sqrt{E(1+u_2'^{2})}]'=0. \tag{6.3}$$

For a right circular cylinder of radius $1/2\|H\|$ (a product of circles $S^1(1/2\|H\|)xS^1(\rho)$ with $\rho=\infty$) we have

$$(L^{3}_{ij})=\begin{bmatrix}2\|H\| & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad (L^{4}_{ij})=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}. \tag{6.4}$$

Then, the equation (6.3) immediately gives

$$\frac{du_2}{du_1}\frac{d^2u_2}{du_1^2}=0, \tag{6.5}$$

for $\alpha=3$, since $L^{3}_{11}=2\,\|H\|$ = constant and E=1. Whence we deduce

$$u_2=C_1u_1+C_2, \tag{6.6}$$

which give the circular helices.

Thus, D-lines in the mean curvature normal direction are the circular helices and all lines are D-lines in the e_4 direction.

For the case of a Clifford flat torus given by (5.1) (a product of circles $S^1(\frac{1}{2}) \times S^1(\frac{1}{2})$) we have

$$(L^3_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad (L^4_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.7}$$

Then, all lines are D-lines in the mean curvature normal direction by (6.3). Since (6.3) reduces to (6.5) for $\alpha=4$, D-lines in the e_4 direction are circular helices.

The Clifford torus may be considered as lying in the 3-sphere of radius 1 which is itself immersed in E^4. A moment's reflection and a glance at (6.7) will show that the mean curvature vector of the Clifford torus in E^4 is the mean curvature vector of $S^3(1) \to E^4$. Consequently, e_4 in the framing used for (6.7) is normal to the Clifford torus that is tangent to $S^3(1)$. We have thus shown that the Clifford torus is a minimal surface in $S^3(1)$.

For the right circular cylinder, the fact that it's geodesic are also D-lines is analogue of the fact for the hypersurfaces in E^3-and the same is true for the sphere-namely : the only surfaces all of whose geodesic are also D-lines are the sphere and the cylinder of revolution. The proof is based on the Laguerre formula and we refer the reader to the extremely elegant work of Şemin [9] for surfaces in E^3. In an earlier paper [5], we defined Laguerre lines of the surfaces of parallel mean curvature in four dimensional manifolds of constant curvature and rewriting the Laguerre formula form the classical point of view, naturally generalizes this fact. We have investigated Laguerre lines of the surfaces of parallel mean curvature in arbitrary dimensional manifolds of constant curvature separately [6].

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