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ON SOME GENERALIZED CESÁRO DIFFERENCE SEQUENCE SPACES

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Abstract : In this paper,we have defined generalized Cesáro difference sequence spaces $C_p(\Delta^m)$, $1 \le p < \infty$, *and* $C_\infty(\Delta^m)$ *and investigated some properties of these spaces and compute their Köthe-Toeplitz duals where* $m \in N$. *Further, we have determined the matrices of classes* $(E, C_p(\Delta^m))$ *and* $(E, C_\infty(\Delta^m))$ *where E denotes one of the sequence spaces* l_∞ *and c namely the linear spaces of bounded and convergent sequences, respectively. This study generalizes some results of Ng and Lee [4] and Orhan [5] in special cases.*

1.Introduction

Orhan [5] defined the Cesáro difference sequence spaces

$$C_p = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p < \infty, \quad 1 \le p < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_k) : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty, \quad n \ge 1 \right\}$$

and showed that the inclusion

$$Ces_p \subset X_p \subset C_p$$

is strict for $1 \le p < \infty$, where $\Delta x = (x_k - x_{k+1}), (k = 1,2,...)$ and Ces_p and X_p are sequence spaces defined by

$$Ces_p = \left\{ x = (x_k) : \|x\|_{p_1} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}} < \infty \quad , 1 \le p < \infty \right\},$$

$$X_p = \left\{ x = (x_k) : \|x\|_{p2} = \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{\frac{1}{p}} < \infty \quad , 1 \le p < \infty \right\}$$

respectively ([6],[4]).Further,the inclusion $l_p \subset Ces_p \subset X_p \subset C_p$ is also strict for $1<p<\infty$, where

$$l_p = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |x_k|^p < \infty, \ 1 \le p < \infty \right\} .$$

The matrix transformations on Cesáro sequence spaces of a non-absolute type are given in [3]. Et and Çolak [1] defined the sequence spaces

$$c(\Delta^m) = \{ x = (x_k) : \Delta^m x \in c \}$$

$$l_\infty(\Delta^m) = \{ x = (x_k) : \Delta^m x \in l_\infty \}$$

and showed that these are Banach spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x_k\|_\infty$$

Now we define

$$C_p(\Delta^m) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right|^p < \infty, \quad 1 \le p < \infty \right\}$$

and

$$C_\infty(\Delta^m) = \left\{ x = (x_k) : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right|^p < \infty, \quad n \ge 1 \right\}$$

where $m \in N = \{1.2...\}$, the set of positive integers, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

It is trivial that $C_p(\Delta^m)$ and $C_\infty(\Delta^m)$ are linear space.

Throughout the paper we write $\lim_n$ for $\lim\limits_{n \to \infty}$.

Theorem 1.1: $C_p(\Delta^m)$ is a Banach space for $1 \le p < \infty$ normed by

$$\|x\|_p = \sum_{i=1}^{m} |x_i| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right|^p \right)^{\frac{1}{p}} \tag{1}$$

and $C_\infty(\Delta^m)$ is a Banach space normed by

$$\|x\|_\infty = \sum_{i=1}^{m} |x_i| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right| \tag{2}$$

Proof. It is a routine verification that $C_\infty(\Delta^m)$ is a normed space normed by (2). To show that $C_\infty(\Delta^m)$ is complete, let (x^s) be a Cauchy sequence in $C_\infty(\Delta^m)$, where $x^s = (x_i^s) = (x_1^s, x_2^s, \ldots) \in C_\infty(\Delta^m)$ for each $s \in N$. Then

$$\|x^s - x^t\|_\infty = \sum_{i=1}^{m} |x_i^s - x_i^t| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k^s - x_k^t) \right| \to 0$$

as $s, t \to \infty$. Hence we obtain

$$|x_k^s - x_k^t| \to 0$$

as $s, t \to \infty$, for each $k \in N$. Therefore $(x_k^s) = (x_k^1, x_k^2, \ldots)$ is a Cauchy sequence in **C**, the set of complex numbers. Since C is complete, it is convergent.

$$\lim_s x_k^s = x_k$$

say, for each $k \in \mathbf{N}$. Since (x^s) is a Cauchy sequence, for each $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that $\|x^s - x^t\|_\infty < \varepsilon$ for all $s,t \geq N$. Hence

$$\sum_{i=1}^{m} |x_i^s - x_i^t| \leq \varepsilon \quad \text{and} \quad \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k^s - x_k^t) \right| \leq \varepsilon$$

for all $k \in \mathbf{N}$ and for all $s,t \geq N$. So we have

$$\lim_t \sum_{i=1}^{m} |x_i^s - x_i^t| = \sum_{i=1}^{m} |x_i^s - x_i| \leq \varepsilon$$

and

$$\lim_t \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k^s - x_k^t) \right| = \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k^s - x_k) \right| \leq \varepsilon$$

for all $s \geq N$. This implies that $\|x^s - x\|_\infty < 2\varepsilon$ for all $s \geq N$, that is, $x^s \to x$ as $s \to \infty$ where $x=(x_k)$.Since

$$\left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right| = \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k + x_k^N - x_k^N) \right|$$

$$\leq \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k^N \right| + \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m (x_k^N - x_k) \right|$$

$$\leq \|x^N - x\|_\infty + \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k^N \right| < \infty$$

we obtain $x \in C_\infty(\Delta^m)$. Therefore $C_\infty(\Delta^m)$ is a Banach space. In the same way it can be shown that $C_p(\Delta^m)$ is a Banach space with norm (1).

Furthermore $x \in C_p(\Delta^m)$ if and only if $\|x\|_p < \infty$, $1 \le p \le \infty$. Since $C_p(\Delta^m)$ $(1 \le p \le \infty)$ is a Banach space with continuous coordinates, that is, $\|x^s - x\|_p \to 0$ implies $|x^s - x| \to 0$ for each $k \in N$, as $s \to \infty$, it is a BK-space.

If we take m=1 and m=0 in Theorem 1.1 we have the following results, respectively.

Corollary 1.2 ([5]): The space C_p $(1 \le p \le \infty)$ is a Banach space.

Corollary 1.3 ([4]): The space X_p $(1 \le p \le \infty)$ is a Banach space.

Now let us define the operator

$$s : C_p(\Delta^m) \to C_p(\Delta^m), x \to sx = (0,0,\ldots x_{m+1}, x_{m+2},\ldots)$$

It is clear that s is a bouded linear operator on $C_p(\Delta^m)$. Furthermore the set

$$s(C_p(\Delta^m)) = sC_p(\Delta^m) = \{x = (x_k) : x \in C_p(\Delta^m), \quad x_1 = x_2 = \ldots = x_m = 0\}$$

is a subspace of $C_p(\Delta^m)$, ($1 \le p \le \infty$).

Now we give some inclusion relations between these sequence spaces.

Theorem 1.4: If $1 \le p < q$, then $C_p(\Delta^m) \subset C_q(\Delta^m)$

Proof: The inequality

$$(\sum_{k=1}^{n} |a_k|^q)^{\frac{1}{q}} \le (\sum_{k=1}^{n} |a_k|^p)^{\frac{1}{p}}, \quad (0 < p < q)$$

[5] gives the proof.

Theorem 1.5 : The inclusion $C_p(\Delta^{m-1}) \subset C_p(\Delta^m)$, $1 \le p \le \infty$, is strict.

Proof.: Let $x = (x_k) \in C_p(\Delta^{m-1})$, $1 \le p < \infty$. Then

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Delta^m x_k\right| \le \left|\frac{1}{n}\sum_{k=1}^{n} \Delta^{m-1} x_k\right| + \left|\frac{1}{n}\sum_{k=1}^{n} \Delta^{m-1} x_{k+1}\right|$$

It is known that, for $1 \le p < \infty$,

$$|a+b|^p \le 2^p(|a|^p + |b|^p)$$

Hence, for $1 \le p < \infty$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Delta^m x_k\right|^p \le M\left\{\left|\frac{1}{n}\sum_{k=1}^{n} \Delta^{m-1} x_k\right|^p + \left|\frac{1}{n}\sum_{k=1}^{n} \Delta^{m-1} x_{k+1}\right|^p\right\}$$

where $M=2^p$. Then , for each positive integer r, we get

$$\sum_{n=1}^{r}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^m x_k\right|^p \leq M\left\{\sum_{n=1}^{r}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^{m-1} x_k\right|^p + \sum_{n=1}^{r}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^{m-1} x_{k+1}\right|^p\right\}$$

Now, as $r \to \infty$

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^m x_k\right|^p \leq M\left\{\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^{m-1} x_k\right|^p + \sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta^{m-1} x_{k+1}\right|^p\right\} < \infty$$

Thus $C_p(\Delta^{m-1}) \subset C_p(\Delta^m)$ $(1 \leq p < \infty)$. The inclusion is strict since the sequence $x = (k^{m-1})$, for example, belongs to $C_p(\Delta^m)$, but does not belong to $C_p(\Delta^{m-1})$ for $1 \leq p < \infty$. Similarly, it can be easily shown that $C_\infty(\Delta^{m-1}) \subset C_\infty(\Delta^m)$. To see that $C_\infty(\Delta^{m-1}) \neq C_\infty(\Delta^m)$, we define the sequence (x_k) by $x_k=k^m$, $(k=1,2,\ldots)$. Then (x_k) is a member of $C_\infty(\Delta^m)$, but not a member of $C_\infty(\Delta^{m-1})$. Now $c(\Delta^m) \subset l_\infty(\Delta^m) \subset C_\infty(\Delta^m)$ and the inclusion is strict since the sequence (x_k) belongs to $C_\infty(\Delta^m)$, but does not belong to $l_\infty(\Delta^m)$, where

$$\Delta^m x_k = \begin{cases} \sqrt{k}, & k = n^2 \\ & \qquad , \quad n=1,2,\ldots \\ 0, & k \neq n^2 \end{cases}$$

Note that $C_p(\Delta^m)$ and $c(\Delta^m)$, overlap but neither one contains the other. Actually the sequence (x_k) by $x_k=k^m$, is an element of $c(\Delta^m)$, but is not an element of $C_p(\Delta^m)$. Moreover the sequence $(x_k)=((-1)^k)$, $(k=1,2,\ldots)$ belongs to $C_p(\Delta^m)$, but does not belong to $c(\Delta^m)$.

Remark : $C_p(\Delta^m)$ $(1 \leq p \leq \infty)$ need not to be sequence algebra. We give a counter example ($m \geq 2$). Let $x=(k)$, $y=(k^{m-1})$. Clearly $x,y \in C_p(\Delta^m)$ $(1 \leq p < \infty)$, $x.y \notin C_p(\Delta^m)$. If we define

$$O_p(\Delta^m) = \left\{x = (x_k) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|\Delta^m x_k\right|\right)^p < \infty, \ 1 \leq p < \infty\right\}$$

$$O_\infty(\Delta^m) = \left\{x = (x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n}\left|\Delta^m x_k\right| < \infty, \quad n \geq 1\right\}$$

then these spaces are normed spaces under the following norms respectively.

$$\|x\|_{p,} = \sum_{i=1}^{m}|x_i| + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|\Delta^m x_k\right|\right)^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|x\|_{\infty 1} = \sum_{i=1}^{m} |x_i| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_k| \right)$$

Clearly $O_p(\Delta^m) \subset C_p(\Delta^m)$, $1 \le p \le \infty$. On the other hand, it is easily seen that $O_p(\Delta^{m-1}) \subset O_p(\Delta^m)$, $1 \le p \le \infty$.

II. Dual Spaces

In this section we give Köthe-Toeplitz duals of $C_\infty(\Delta^m)$ and $O_\infty(\Delta^m)$

Lemma 2.1 : $x \in sC_\infty(\Delta^m)$ implies $\sup_k k^{-1} |\Delta^{m-1} x_k| < \infty$.

Proof is trivial.

Lemma 2.2 *: ([1]).* $\sup_k k^{-1} |\Delta^{m-1} x_k| < \infty$ implies $\sup_k k^{-m} |x_k| < \infty$.

Corollary 2.3 *:* $x \in sC_\infty(\Delta^m)$ implies $\sup_k k^{-m} |x_k| < \infty$.

Definition 2.4 *: ([2])* Let X be a sequence space and define

$$X^\alpha = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty \quad \text{for all} \quad x \in X \right\},$$

Then X^α is called the α- dual spaces of X. X^α is also called Köthe-Toeplitz dual space. It is easy to show that $\emptyset \subset X^\alpha$. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$. It is clear that $X \subset X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$ then X is called a α-space. In particular, an α-space is called a Köthe space or a perfect sequence space.

Lemma 2.5 *:* $\left[sC_\infty(\Delta^m) \right]^\alpha = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}$.

Proof *:* Let $U_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}$. If $a \in U_1$, then

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} k^m |a_k| (k^{-m} |x_k|) < \infty$$

for each $x \in sC_\infty(\Delta^m)$, by Corollary 2.3. Hence $a \in \left[sC_\infty(\Delta^m) \right]^\alpha$

Let $a \in \left[sC_\infty(\Delta^m) \right]^\alpha$. Then $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for each $a \in sC_\infty(\Delta^m)$.

For the sequence $x = (x_k)$, defined by

$$x_k = \begin{cases} 0, & k \le m \\ k^m, & k > m \end{cases} \qquad (2.1)$$

we may write

$$\sum_{k=1}^{\infty} |k^m a_k| = \sum_{k=1}^{m} |k^m a_k| + \sum_{k=1}^{\infty} |k^m a_k| < \infty$$

This implies $a \in U_l$.

Theorem 2.6. $[sC_\infty(\Delta^m)]^\alpha = [C_\infty(\Delta^m)]^\alpha$

Proof. Since $sC_\infty(\Delta^m) \subset C_\infty(\Delta^m)$, then $[C_\infty(\Delta^m)]^\alpha \subset [sC_\infty(\Delta^m)]^\alpha$.

Let $a \in [sC_\infty(\Delta^m)]^\alpha$ and $x \in C_\infty(\Delta^m)$. If we take the sequence $x=(x_k)$,

$$x_k = \begin{cases} x_k, & k \le m \\ x'_k, & k > m \end{cases}$$

where $x'=(x'_k) \in sC_\infty(\Delta^m)$. Then we may write

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{m} |a_k x_k| + \sum_{k=1}^{\infty} |a_k x'_k| < \infty$$

This implies that $a \in [C_\infty(\Delta^m)]^\alpha$.

Theorem 2.7 : $[O_\infty(\Delta^m)]^\alpha = [C_\infty(\Delta^m)]^\alpha$.

Proof is trivial.

Theorem 2.8 : For $X = O_\infty(\Delta^m)$ or $C_\infty(\Delta^m)$

$$[X]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-m} |a_k| < \infty\}$$

Proof is trivial.

Corollary 2.9 : X is not perfect.

III.Matrix Transformations

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} (n,k =1 ,2, ...) and X,Y be two subsets of the space of complex sequences we write formally

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k \qquad (n = 1,2, ...), \qquad (3)$$

and say that the matrix $A = (a_{nk})$ defines a matrix transformations from X into Y and it is denoted by writing $A \in (X,Y)$. If each series in (3) converges and $((A_n(x)) \in Y$ whenever

$(x_k) \in X$. Furthermore, let (X,Y) be the set of all infinite matrices $A = (a_{nk})$ which map the sequence space X into the sequence space Y. We now determine the matrices of classes $(E, C_p(\Delta^m))$, $1 \le p \le \infty$, where E denotes one of the sequence spaces l_∞, all bounded complex sequences, and c, all convergent complex sequences.

Theorem 3.1 : $A \in (E, C_p(\Delta^m))$, $1 \le p < \infty$, if and only if

i) $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, for each n

ii) $B \in (E, l_p)$

where $B = (b_{ik}) = \frac{1}{i}(\Delta^{m-1} a_{1k} - \Delta^{m-1} a_{i+1,k})$.

Proof : Sufficiency is trivial.

Necessity: Suppose that $A = (a_{nk})$ maps E into $C_p(\Delta^m)$, $(1 \le p < \infty)$ then the series

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

is convergent for each n and for all $x \in E$ and $(A_n(x)) \in C_p(\Delta^m)$. Since $E^\beta = l_1$ for $E = l_\infty$ or c, then we get (i). Furthermore, since $(A_n(x)) \in C_p(\Delta^m)$,

$$\sum_{i=1}^{\infty} \left| \frac{1}{i} \sum_{n=1}^{i} \Delta^m A_n(x) \right|^p = \sum_{i=1}^{\infty} \left| \frac{1}{i} (\Delta^{m-1} A_1(x) - \Delta^{m-1} A_{i+1}(x) \right|^p < \infty$$

for all $x \in E$ anf for $1 \le p < \infty$. Whereas

$$\frac{1}{i}(\Delta^{m-1} A_1(x) - \Delta^{m-1} A_{i+1}(x)) = \sum_{k=1}^{\infty} \frac{1}{i} (\Delta^{m-1} a_{1k} - \Delta^{m-1} a_{i+1,k}) x_k$$

for $x \in E$. If we now set

$$B_i(x) = \sum_{k=1}^{\infty} \frac{1}{i} (\Delta^{m-1} a_{1k} - \Delta^{m-1} a_{i+1,k}) x_k$$

Then $(B_i(x)) \in l_p$, $(1 \le p < \infty)$. So that $B \in (E, l_p)$ where

$$B = (b_{ik}) = \frac{1}{i}(A^{m-1} a_{1k} - \Delta^{m-1} a_{i+1,k})$$

for all i, k. Hence the necessity is proved.

Theorem 3.2 : $A \in (E, C_\infty(\Delta^m))$, if and only if

i) $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, for each n

ii) $B \in (E, l_\infty)$

where $B = (b_{ik}) = \frac{1}{i}(\Delta^{m-1} a_{1k} - \Delta^{m-1} a_{i+1,k})$.

Proof is trivial.

Corollary 3.3 ([5]) : $A \in (E, C_p)$, $1 \leq p < \infty$, if and only if

i) $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, for each n

ii) $B \in (E, l_p)$

where $B = (b_{ik}) = \frac{1}{i}(a_{ik} - a_{i+1,k})$ for all i,k.

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