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ON PROJECTIVE COLLINEATION IN FINSLER SPACE II

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Recently S. P. Singh ([7], [8])[1] has derived curvature collineation in Finsler spaces. F. M. Meher [6] has discussed projective motion in symmetric Finsler space. The object of this paper is to study R*- projective curvature collineation in Finsler space. Some special cases are also discussed at the end. The notations used in the sequel are due to E.Cartan [1] and H. Rund [4].

1. INTRODUCTION

We consider an n-dimensional Finsler space Fn with connection parameters $\Gamma^{*i}{}_{jk}(x,\dot{x})$[2] which is homogeneous of degree zero in $\dot{x}^i$.

The covariant derivatives of the scalar $f(x,\dot{x})$ and a tensor $T^i{}_{jk}(x,\dot{x})$ are given by

$$(1.1)\qquad f_{;j} = f_{,j} - f\big|_a \Gamma^{*a}{}_{bj}\dot{x}^b$$

$$(1.2)\qquad T^i{}_{jk;l} = T^i{}_{jk,l} - T^i{}_{jk}\big|_a \Gamma^{*a}{}_{bl}\dot{x}^b + T^a{}_{jk}\Gamma^{*i}{}_{al} - T^i{}_{ak}\Gamma^{*a}{}_{jl} - T_{ja}{}^i\Gamma^{*a}{}_{kl},$$

respectively, where a comma and a vertical stroke denote the partial derivatives of the function with respect to x^i and $\dot{x}^i$ respectively.

The corresponding curvature tensor field in Fn is defined by

$$(1.3)\qquad R^{*i}{}_{jkl} = \left(\Gamma^{*i}{}_{jk,l} - \Gamma^{*i}{}_{jk}\big|_a \Gamma^{*a}{}_{bl}\dot{x}^b\right) - \left(\Gamma^{*i}{}_{jl,k} - \Gamma^{*i}{}_{jl}\big|_a \Gamma^{*a}{}_{bk}\dot{x}^b\right) + \Gamma^{*a}{}_{jk}\Gamma^{*i}{}_{al} - \Gamma^{*a}{}_{jl}\Gamma^{*i}{}_{ak}.$$

The commutation formulae involving the connection parameters $\Gamma^{*i}{}_{jk}$ and the curvature tensor field $R^{*i}{}_{jkh}$ are given by

$$(1.4)\qquad T^i{}_{jk;l}\big|_m - T^i{}_{jk;m}\big|_l = T^a{}_{jk}\Gamma^{*i}{}_{al}\big|_m - T^i{}_{ak}\Gamma^{*a}{}_{jl}\big|_m - T^i{}_{ja}\Gamma^{*a}{}_{kl}\big|_m - T^i{}_{jk}\big|_a\Gamma^{*a}{}_{bl}\big|_m\dot{x}^b,$$

$$(1.5)\qquad T^i{}_{jk;l;m} - T^i{}_{jk;m;l} = T^a{}_{jk}R^{*i}{}_{alm} - T^i{}_{ak}R^{*a}{}_{jlm} - T^i{}_{ja}R^{*a}{}_{klm} - T^i{}_{jk}\big|_a R^{*a}{}_{blm}\dot{x}^b.$$

The Lie-derivative of the tensor $T^i{}_{jk}$ and the connection parameters $\Gamma^{*i}{}_{jk}$, defined by the infinitesimal transformation

$$(1.6)\qquad \bar{x}^i = x^i + \xi^i(x)dt,$$

are characterised by

$$(1.7)\qquad \pounds T^i{}_{jk} = T^i{}_{jk;a}\xi^a + T^i{}_{jk}\big|_a \xi^a{}_{;b}\dot{x}^b - T^a{}_{jk}\xi^i{}_{;a} + T^i{}_{ak}\xi^a{}_{;j} + T^i{}_{ja}\xi^a{}_{;k}$$

and

$$(1.8)\qquad \pounds\Gamma^{*i}{}_{jk} = \xi^i{}_{;j;k} + R^{*i}{}_{jkl}\xi^l + \Gamma^{*i}{}_{jk}\big|_a \xi^a{}_{;b}\dot{x}^b,$$

respectively.

The processes of Lie –differentiation and other differentiations are connected by

$$(1.9)\qquad \left(\pounds T^i{}_{jk}\right)\big|_l - \pounds\left(T^i{}_{jk}\big|_l\right) = 0,$$

(1.10) $$\left(\pounds T^{i}{}_{jk}\right)_{;l} - \left(T^{i}{}_{jk;l}\right) = -T^{a}{}_{jk}\pounds\Gamma^{*i}{}_{al} + T^{i}{}_{ak}\Gamma^{*a}{}_{jl} + T^{i}{}_{ja}\pounds\Gamma^{*a}{}_{kl} + T^{i}{}_{jk}\big|_{a}\pounds\Gamma^{*a}{}_{bl}\dot{x}^{b}.$$

Also the Lie - derivative of the of the curvature tensor $R^{*i}{}_{jkl}$ is expressed as

(1.11) $$\left(\pounds\Gamma^{*i}{}_{jk}\right)_{;l} - \left(\pounds\Gamma^{*i}{}_{jl}\right)_{;k} = \pounds\ R^{*i}{}_{jkl} + \Gamma^{*i}{}_{jk}\big|_{a}\pounds\ \Gamma^{*a}{}_{bl}\dot{x}^{b} - \Gamma^{*i}{}_{jl}\big|_{a}\pounds\ \Gamma^{*a}{}_{bk}\dot{x}^{b}.$$

The infinitesimal transformation (1.6) defines a projective motion if it transforms the system of geodesies into those of geodesies. The necessary and sufficient condition that (1.6) be a projective motion in Fn is that the Lie –derivative of the connection coefficients $\Gamma^{*i}{}_{jk}$ have the form

(1.12) $$\pounds\Gamma^{*i}{}_{jk} = 2\delta^{i}{}_{(j}p^{*}{}_{k)} + \dot{x}^{i}p^{*}{}_{jk},^{3)}$$

where

(1.13) $$p^{*}{}_{k} = \dot{\partial}_{k}p^{*},\ \ p^{*}{}_{jk} = \dot{\partial}_{j}p^{*}{}_{k}$$

for some homogeneous scalar function $p^{*}(x,\dot{x})$ of degree one in $\dot{x}^{i}$.

For homogeneity of $p^{*}{}_{k}$ and $p^{*}{}_{jk}$, they satisfy

(1.14) $$p^{*}{}_{k}\dot{x}^{k} = p^{*},\ p^{*}{}_{jk}\dot{x}^{k} = 0.$$

2. PROJECTIVE CURVATURE COLLINEATION

Definition 2.1 In a Finsler space Fn , if the curvature tensor $R^{*i}{}_{jkh}$ satisfies the relation

(2.1) $$\pounds R^{*i}{}_{jkh} = 0,$$

where £ represents Lie-derivative defined by the infinitesimal transformation (1.6), which admits projective motion , then the transformation (1.6) is called projective R^{*}-curvature collineation.

The infinitesimal transformation (1.6) is called a projective motion in Fn ,if the Lie –derivative of $\Gamma^{*i}{}_{jk}$ satisfies the relation (1.12). Applying (1.12) in the equation (1.11), it gives

(2.2) $$2\delta^{i}{}_{j}p^{*}_{[k;l]} + 2\delta^{i}_{[k}p^{*}_{|j|;l]} + 2\dot{x}^{i}p^{*}_{[k|j|;l]} = \pounds R^{*i}_{jkl} + \Gamma^{*i}_{jk}\big|_{a}\left(2\delta^{i}_{(b}p^{*}_{l)} + \dot{x}^{a}p^{*}_{bl}\right)\dot{x}^{b} - \Gamma^{*i}_{jl}\big|_{a}\left(2\delta^{a}_{(b}p^{*}_{k)} + \dot{x}^{a}p^{*}_{bk}\right)\dot{x}^{b},$$

where the index within two parallel bars is unaffected when we consider skew symmetric part.

If we take

(2.3) $$\delta^{i}_{j}p^{*}_{[k;l]} + \delta^{i}_{[k}p^{*} + \dot{x}^{i}p^{*}_{[k|j|;l(\ ]]} = 0,$$

then the equation (2.2) assumes the form

(2.4) $$\pounds R^{*}_{jkl} = \Gamma^{*i}_{jl}\big|_{a}\left(2\delta^{a}_{(b}p^{*}_{k)}\right)\dot{x}^{b} - \Gamma^{*i}_{jk}\big|_{a}\left(\delta^{a}_{(b}p^{*}_{l)} + \dot{x}^{a}p^{*}_{bl}\right)\dot{x}^{b}.$$

In view of (1.14) and the homogeneity of connection coefficients. equation (2.4) reduces to (2.1) provided the space Fn is an affinely connected space.

Conversely, if (2.1) is true, the equation (2.2) assumes the form

(2.5)

By virtue of (1.14) and homogeneity property of the connection coefficient Γ^{*i}_{jk}, the equation (2.5) reduces to (2.3) when the space in consideration is also affinely connected space.

We thus state

Theorem 2.1 The necessary and sufficient condition for the infinitesimal transformation (1.6) to be projective R^*-curvature collineation in Fn, is that the scalar function $p^*(x,\dot{x})$ satisfies the relation (2.3) and Fn is an affinely connected space.

From the equation (1.12) and (1.13), it is clear that the vanishing of the scalar function p_j^*, that is $p_j^* = 0$, is the necessary and sufficient condition for the projective motion to be an affine motion. This condition also identically satisfies the equation (2.3).

Thus we conclude

Corollary 2.1 In a Finsler space Fn, every projective R^*-curvature collineation is an affine motion.

In view of the commutation formula (1.9) for the curvature tensor R^{*i}_{jkl}, we have

$$(2.6) \qquad \left(\pounds R^{*i}_{jkh}\right)\big|_l = \pounds\left(R^{*i}_{jkh}\big|_l\right)$$

Applying (2.1) in equation (2.6), we obtain

$$(2.7) \qquad \pounds\left(R^{*i}_{jkh}\big|_l\right) = 0.$$

Hence we have

Lemma 2.1 In a Finsler space Fn, which admits the projective R^*-curvature collineation, the partial derivative of the curvature tensor R^{*i}_{jkh} is Lie –invariant.

Applying the commutation formula (1.5) to the curvature tensor R^{*i}_{jkh}, we get

$$(2.8) \qquad R^{*i}_{jkh;l;m} - R^{*i}_{jkh;m;l} = R^{*a}_{jkh} R^{*i}_{alm} - R^{*i}_{akh} R^{*a}_{jlm} - R^{*i}_{jah} R^{*a}_{klm} - R^{*i}_{jkh}\big|_a R^{*a}_{blm}\dot{x}^h - R^{*i}_{jka} R^{*a}_{hlm}.$$

Taking the Lie –operator of both sides of (2.8) and using (2.1) and lemma 2.1, we find

$$(2.9) \qquad \pounds R^{*i}_{jkh;l;m} = \pounds R^{*i}_{jkh;m;l},$$

if $\qquad \pounds\dot{x}^i = 0$.

Thus, we state

Theorem 2.2 In a Finsler space Fn, which admits the projective R^*-curvature collineation, the identity (2.9) holds good provided $\pounds\dot{x}^i = 0$.

Using the identity (1.10) for the curvature tensor field R^{*i}_{jkh}, we get

$$(2.10) \qquad \pounds\left(R^{*i}_{jkh;l}\right) = \left[2\delta^i_{(a}p^*_{l)} + \dot{x}^i p^*_{al}\right]R^{*a}_{jkh} - \left[2\delta^a_{(j}p^*_{l)} + \dot{x}^a p^*_{jl}\right]R^{*i}_{akh} - \left[2\delta^a_{(k}p^*_{l} + \dot{x}^a p^*_{kl}\right]R^{*i}_{jah}$$
$$- \left[2\delta^a_{(h}p^*_{l)} + \dot{x}^a p^*_{hl}\right]R^{*i}_{jah} - \left[2\delta^a_{(h}p^*_{l)} + \dot{x}^a p^*_{hl}\right]R^{*i}_{jka} - \left[2\delta^a_{(b}p^*_{l)} + \dot{x}^a p^*_{bl}\right]R^{*i}_{jkh}\dot{x}^h$$

in view of (1.12) and (2.1).

Since the vanishing of the scalar function p_j^*, is a necessary and sufficient condition for projective motion to be an affine motion, the equation (2.10) yields

$$(2.11) \qquad \pounds\left(R^{*i}_{jkh;l}\right) = 0.$$

Accordingly we state

Theorem 2.3 When a projective R^*- curvature collineation admitted in a Finsler space Fn, becomes a motion, the covariant derivative of the curvature tensor field R^{*i}_{jkh} is Lie - invariant.

Hiramatu [2] has established that the necessary and sufficient condition for an infinitesimal transformation to be homothetic one is that the relation

$$(2.12) \qquad \pounds g_{ij} = g_{ia}\xi^a_{;j} + g_{aj}\xi^a_{;i} + 2C_{ija}\xi^a_{;b}\dot{x}^h = 2cg_{ij},$$

where c is a constant, holds good. Any solution of (2.12) satisfies the relation $£\Gamma^{*i}_{jk} = 0$. Also the necessary and sufficient condition for $£\Gamma^{*i}_{jk} = 0$ to be true is $p^*_j = 0$, which identically satisfies the relation (2.3).

Thus we state

Theorem 2.4 Every homothetic transformation admitted in a Finsler space Fn is a projective R^* - curvature collineation .

The non-flat Finsler space Fn , whose curvature tensor field R^{*i}_{jkh} satisfies the relation

$$R^{*i}_{jkh;l} = K_l R^{*i}_{jkh} \tag{2.13}$$

for a non-zero vector K_l is called recurrent Finsler space [5]. We denote this recurrent Finsler space by F^*_n. The vector K_l is called recurrence vector.

Taking covariant derivative of (2.13) with respect to x^m, we obtain

$$R^{*i}_{jkh;l;m} = \left(K_{l;m} + K_l K_m\right) R^{*i}_{jkh}, \tag{2.14}$$

which yields

$$R^{*i}_{jkh;l;m} - R^{*i}_{jkh;m;l} = K_{[l;m]} R^{*i}_{jkh} \quad . \tag{2.15}$$

Taking the Lie – derivative of both sides of the equation (2.15), we get

$$£K_{[l;m]} = 0 \tag{2.16}$$

in view of (2.1) and (2.9) since F^*_n is non-flat.

Hence we state

Theorem 2.5 In a recurrent Finsler space F^*_n, which admits the projective R* - curvature collineation, the recurrence vector K_l satisfies the relation

$$£K_{[l;m]} = 0 .$$

3. SPECIAL CASES

(a) Contra Field

In a Finsler space Fn , if the vector field $\xi^i(x)$ satisfies the relation

$$\xi^i_{;j} = 0, \tag{3.1}$$

the vector field determines a contra field . In this case, we consider the projective R* - curvature collineation in the form

$$\bar{x}^i = x^i + \xi^i(x)dt, \quad \xi^i_{;j} = 0. \tag{3.2}$$

Applying (1.12) and (3.2), the equation (1.8) yields

$$R^{*i}_{jkh}\xi^h = 2\delta^i_{(j} p^*_{k)} + \dot{x}^i p^*_{jk}. \tag{3.3}$$

Differentiating the above equation covariantly with respect to x^l and using (3.1), we get

$$R^{*i}_{jkh;l}\xi^h = \delta^i_j p^*_{k;l} + \delta^i_k p^*_{j;l} + \dot{x}^i p^*_{jk;l}, \tag{3.4}$$

which yields

$$R^{*i}_{jh[k;l]}\xi^h = 0 \tag{3.5}$$

in view of (2.3). Accordingly we state

Theorem 3.1 In a Finsler space Fn , which admits the projective R* - curvature collineation , if the vector field $\xi^h(x)$ spans a contra field, the relation

$$R^{*i}_{jh[k;l]}\xi^h = 0$$

holds good.

Taking the covariant derivative of (3.3) with respect to x^l, we obtain

$$(3.6)\qquad K_l R^{*i}_{jkh}\xi^h = \delta^i_j p^*_{k;l} + \delta^i_k p^*_{j;l} + \dot{x}^i p^*_{jk;l}$$

in view of (2.11) and (3.1).

Applying (2.3) in the above equation, it gives

$$(3.7)\qquad R^{*i}_{jh[l}K_{l]}\xi^h = 0,$$

since R^{*i}_{jkh} is skew symmetric in the last two covariant indices. Thus we have

Theorem 3.2 In a recurrent Finsler space F^*_n, which admits the projective R* - curvature collineation, if the vector field $\xi^i(x)$ spans a contra field, the relation (3.7) is true.

(b) Concurrent Field

In a Finsler space Fn , if the vector field $\xi^i(x)$ satisfies the relation

$$(3.8)\qquad \xi^i_{;l} = \lambda\delta^i_l,$$

where λ is a non-zero constant, then the vector field $\xi^i(x)$ determines a concurrent field. We shall in this case consider projective R* - curvature collineation of the form

$$(3.9)\qquad \bar{x}^i = x^i + \xi^i(x)dt, \qquad \xi^i_{;j} = \lambda\delta^i_j.$$

Using the equation $(1.8),(1.12)$ and (3.8), we obtain relation (3.3).

Differentiating (3.3) covariantly with respect to x^l and noting (2.11) and (3.8), it yields

$$(3.10)\qquad R^{*i}_{jkh;l}\xi^h + \lambda R^{*i}_{jkl} = \delta^i_j p^*_{k;l} + \delta^i_k p^*_{j;l} + \dot{x}^i p^*_{jk;l}.$$

Commuting the indices j and l in (3.10) and using (2.3), we get

$$(3.11)\qquad R^{*i}_{j[k|h|;l]} + \lambda R^{*i}_{j[kl]} = 0.$$

Thus we state

Theorem 3.3 In a Finsler space Fn , which admits the projective R* - curvature collineation, if the vector field $\xi^i(x)$ determines a concurrent field, then the relation (3.10) holds good.

In view of (2.11) and (3.8), the covariant derivative of (3.3) yields

$$(3.12)\qquad K_l R^{*i}_{jkh}\xi^h + \lambda R^{*i}_{jkl} = \delta^i_j p^*_{k;l} + \delta^i_k p^*_{j;l} + \dot{x}^i p^*_{jk;l}.$$

Since the curvature tensor R^{*i}_{jkh} is skew – symmetric in last two covariant indices, the equation (3.12) assumes the form

$$(3.13)\qquad R^{*i}_{jh[k}K_{l]}\xi^h = \lambda R^{*i}_{j[kl]}$$

in view of (2.3)

Hence we state

Theorem 3.4 In a recurrent Finsler space Fn , which admits the projective R* - curvature collineation, if the vector field $\xi^i(x)$ determines a concurrent field, the relation (3.13) is necessary true.

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