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# MATRIX TRANSFORMATION OF $\ell(p,s)$ TO $\ell_\infty(p)$ AND $c_0(p)$

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Abstract: In this paper we have determined necessary and sufficient contidions for an infinite matrix $A=(a_{n,k})$ to transform $l(p,s)$ into $l_\infty(p)$ and $c_0(p)$.

## 1. Introduction

Lat N and C denote the sets of natural numbers and complex numbers, respectively.

X will denote a notrivial complex linear space of elements x, with zero element θ and with paranorm g.A subset G of X is called a fundamental set in X if linear bull (G), the set of all finite linear combinations of elements of G, is dense in X.A. sequence $(b_k)$ of elements of X is said to be a basis in X if for each $x\in X$ there is a unique complex sequence $(\lambda_k)$ suhc that $g\left(x-\sum_{k=1}^{n}\lambda_k b_k\right)\to 0$ $(n\to\infty)$. Thus any basis in X is also a fundamental set in X.

We denote the set of continuous linear functionals on X by $X^*$. A linear functional A on X is an element of $X^*$ if and only if

$$\|A\|_M\equiv\left\{|A(x)|: g(x)\le\frac{1}{M}\right\}<\infty \text{ for some } M>1.$$

If x is a space of complex sequence $x=(x_k)$, then we denote the generalize Köthe-Teoplitz dual of X by $X^+$, i.e.

$$X^+=\left\{(\alpha_k):\sum_k \alpha_k x_k \text{ converges for every } x\in X\right\}.$$

(Throughout $\sum_k$ denotes summation over k from k=1 ot k=∞).

The following a paranormed β-space were defined by Maddox (1974). Let $(X_n)$ be asequence of subsets of X such that $\theta\in X_j$ and such that if $x,y\in X_n$ tben $x\pm y\in X_{n+1}$ for $n\in N$; tehn $(X_n)$ is called an α-space in X. ıf we can write $X=\bigcup_{n=1}^{\infty}X_n$ ,, where $(X_n)$ is an α-sequence in X and each $X_n$ is nowhere dense in X, then X is called α-space; otherwise X is a β-space. Clearly, every α-space is of the first category, whence we see that any complete paranormed space is a β-space.

We now list some sets of complex sequence due to Maddox (1967) and Bulut E. and Çakar Ö (1979). If $p=(p_k)$ is a sequence of strictly positive real numbers, then

$$l_\infty(p)=\left\{x: \sup_k |x_k|^{p_k}<\infty\right\},$$

$$c_0(p)=\left\{x: \lim_k |x_k|^{p_k}=0\right\},$$

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$$l(p,s)=\left\{x:\sum_k k^{-s}|x_k|^{p_k}<\infty,\quad s\geq 0\right\}$$

Now we collect some known results which will be useful in what follows.

Lemma 1. $l(p,s)$ is a linear space if and only if

$0<p_k\leq \sup_k p_k=H<\infty$ (Buiut E. and Çakar Ö. 1979).

Lemma 2. If $0<\inf_k p_k\leq p_k\leq \sup_k p_k=H<\infty$ with $M=\max(H,1)$,

then

$$g(x)=\left\{\sum_k k^{-s}|x_k|^{p_k}\right\}^{1/M}$$

defines a paranorm on $l(p,s)$, $l(p,s)$ is complete under g, and $(e^{(k)})$ is a basis in $l(p,s)$, where $e^{(k)}$ is a sequence with 1 in the k th place and zero elsewhere (Bulut E. and Çakar Ö. 1979).

Theorem 3. (i) If $1<p_k\leq \sup_k p_k=H<\infty$ and $p_k^{-1}+q_k^{-1}=1$ for each $k\in N$, then

$$l(p,s)^{+}=\left\{a=(a_k):\sum_k k^{s(q_k-1)}N^{-q_k/p_k}|a_k|^{q_k},\ s\geq 0,\ \text{for some integer } N>1\right\}$$

(ii) If $0<\inf_k p_k\leq 1$ for all $k\in N$ then

$l(p,s)^{+}=\{a=(a_k):\sup_k k^{s}|a_k|^{p_k}<\infty,\ s\geq 0\}$ (Theorem 1 in Bulut E. and Çakar Ö. 1979).

Theorem 4. If either $1<p_k\leq \sup_k=H<\infty$ for all k, or $0<\inf_k p_k\leq p_k\leq 1$ for all k, then every $A\in l(p,s)^{*}$ may be written as $A(x)\equiv\sum_k\alpha_k x_k$ on $l(p,s)$ for some $(\alpha_k)\in l(p,s)^{+}$, and conversely $A(x)\equiv\sum_k\alpha_k x_k$ defines an element of $l(p,s)^{*}$ for each $(\alpha_k)\in l(p,s)^{+}$ (Theorem 2 in Bulut E. and Çakar Ö. 1979).

Theorem 5. Let X be a paranormed space and let $(A_n)$ be a sequence elements of $X^{*}$, and suppose r is bounded, where r is a sequence of strictly positive real numbers. Then

(1) $\sup_n(\|A\|_M)^{r_n}<\infty$ for some $M>1$

implies

(2) $(A_n(x))\in l_\infty(r)$ for every $x\in X$,

and the converse is true if X is a β-space. (Theorem 1 in Maddox I.J. and Willey M.A.L. 1974).

Theorem 6. Let X be a paranormed space and let $(A_n)$ be a sequence of elements of $X^{*}$.

(i) If X has fundamental set G and if r is bounded, then the fololwing propositions

(3) $(A_n(b))\in c_0(r)$ for every $b\in G$,

(4) $\lim_M \limsup_n (\|A_n\|_M)^{r_n}=0$,

together imply

(5) $(A_n(x))\in c_0(r)$ for every $x\in X$.

(i) If $r_n\to 0$ $(n\to\infty)$ then (4) implies (5).

(iii) Let X be a β-space; tehn (5) implies (4) even if r is unbounded. (Theorem 2 in Maddox I.J. and Willey M.A.L. 1974).

Theorem 7. Let X be a paranormed space and let $(A_n)$ be a sequence of elements of $X^*$ and suppose r is bounded.

(i) If X has fundamental set G, and if there is an $L \in X^*$ such that

$(A_n(b)-L(b)) \in c_0(r)$ for all $b \in G$ and

(6) $\lim_M \lim\sup_n (\|A_n-L\|_M)^{r_n}=0$,

then

(7) $(A_n(x)) \in c(r)$ on X.

(ii) If $r_n \to 0$ $(n \to \infty)$ and if there is an $L \in X^*$ such that (6) holds, then (7) is true.

(iii) If X is a β-space and if (7) is true, then there is an $L \in X^*$ such that (6) holds. (THeorem 3 in Maddox I.J. and Wiliey M.A.L. 1974).

Let Y and Z be sets of sequences. We shal write (Y, Z) for the class of matrices $A=(a_{n,k})$, $n,k=1,2,\ldots$ of complex numbers $a_{n,k}$, such that for each $y=(y_k) \in Y$,

$$A_n(y)=\Sigma_k a_{n,k} y_k$$

converges for each n, and $(A_n(y)) \in Z$. The class (Y,Z) is said to be the set of matrices transforming Y into Z.

**Theorem 8.** (i) If $1<p_k \le \sup_k p_k=H<\infty$ and $p_k^{-1}+q_k^{-1}=1$ for every $k \in N$ then $A \in (l(p,s),l_\infty)$ if and only if there exists an integer $B>1$ such that

$$\sup_k \sum_k k^{s(q_k-1)} B^{-q_k} |a_{n,k}|^{q_k} < \infty .$$

(ii) If $0<\inf_k p_k \le p_k \le 1$ for each $k \in N$, then $A \in (l(p,s),l_\infty)$ if and only if

$\sup_{n,k} k^s |a_{n,k}|^{p_k} < \infty$. (Theorem 3 in Bulut B. and Çakar Ö. 1979)

We shall frequently use the following inequalities. Take $x, y \in C$; if $0<p \le 1$ then

$$|x|^p-|y|^p \le |x+y|^p \le |x|^p+|y|^p,$$

and f $p>1$ and $p^{-1}+q^{-1}=1$ then $|xy| \le |x|^p+|y|^q$.

**2. Matrix Tranfsormations**

In the profs of the following result, as in earlier ones, we may without loss of generality assume that $r_n \le 1$ for all $n \in N$, and we shall do so when convenient.

**Theorm 9.** (i) Let $0<\inf_k p_k \le p_k \le 1$ and $p_k^{-1}+q_k^{-1}=1$ for each $k \in N$, and let r be bounded. Then $A \in (l(p,s),l_\infty(r))$ if and only if

(8) $$\sup_n \left\{\sup_k k^{s/p_k} |a_{n,k}| M^{-1/p_k}\right\}^{r_n} < \infty \text{ for some } M>1.$$

(ii) Let $1<p_k \le \sup_k p_k=H<\infty$ and $p_k^{-1}+q_k^{-1}=1$ for each $k \in N$, and let r be bounded, Then $A \in (l(p,s),l_\infty(r)$ if and only if

(9) $$T(B) \equiv \sup_n \sum_k k^{s(q_k-1)} B^{-q_k/r_n} |a_{n,k}|^{q_k} < \infty \text{ for some } B>1.$$

Proof. Write, for each $x \in (l(p,s), l_\infty(r))$; then for each n, $(a_{n,1}, a_{n,2}, \dots) \in l(p,s)^+$. Also, by Theorem 4, $A_n \in l(p,s)^*$ for each $n \in N$. We show that for each n, $\|A_n\|_M = \sup_k |a_{n,k}| k^{s/p_k} M^{-1/p_k}$ for al M such that $\|A_n\|_M$ is defined. Choose any $n \in N$. First, if M is such that, for some sequence (k(i)) of integers,

$$|a_{n,k(i)}| k(i)^{s/p_{k(i)}} M^{-1/p_{k(i)}} \geq i \text{ for each } i \in N, \text{ then by defining}$$

$$x^{(k(i))} = (M^{-1/p_{k(i)}} \operatorname{sgn} a_{n,k(i)}{}^{s/p_{k(i)}}) e^{(k(i))},$$

i=1,2,..., we see that $\|A_n\|_M$ is underfined. Since $(a_{n,1}, a_{n,2}, \dots) \in l(p,s)^+$ there is an $M_n \geq 1$ such that $|a_{n,k}|^{p_k} k^s \leq M_n$ for all k. Choose $M \geq M_n$. We have if $g(x) = \sum_k k^{-s} |x_k|^{p_k} \leq 1/M$, since $M^{1/p_k} k^{-s/p_k} |x_k| \leq 1$ for all k and since $\sup_k p_k \leq 1$,

$$|A_n(x)| \leq \sum_k |a_{n,k} x_k|$$

$$= \sum_k k^{s/p_k} k^{-s/p_k} |x_k| M^{1/p_k} M^{-1/p_k} |a_{n,k}|$$

$$\leq \sum_k k^{s/p_k} k^{-s} |x_k|^{p_k} M\, M^{-1/p_k} |a_{n,k}|$$

$$\leq M \sup_k (k^{s/p_k} |a_{n,k}| M^{-1/p_k}) (1/M),$$

whence $\|A_n\|_M \leq \sup_k k^{s/p_k} |a_{n,k}| M^{-1/p_k}$. Given $\varepsilon > 0$ we can choose an m such that

$$|a_{n,m}| M^{-1/p_m} m^{s/p_m} > \sup_k |a_{n,k}| M^{-1/p_k} k^{s/p_k} - \varepsilon.$$

Defining $x = (M^{-1/p_m} \operatorname{sgn} a_{n,m} m^{s/p_m}) e^{(m)}$ we have $g(x) \leq 1/M$ and

$$A_n(x) = |a_{n,m}| M^{-1/p_m} m^{s/p_m} > \sup_k |a_{n,k}| M^{-1/p_k} k^{s/p_k} - \varepsilon,$$

whence $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k} k^{s/p_k}$ as required. By the Lemma 2,l (p,s) is complete, so it is a β-space; tuhs by Theorem 5 we must have (8).

Conversel let (8) hold. Then as above it follows that for each n, $A_n \in l(p,s)^*$ with $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k} k^{s/p_k}$ for all M such that $\|A_n\|_M$ is defined. Then using Theorem 5 we obtain $(A_n(x)) \in l_\infty(r)$.

(ii) For the sufficiency, let (9) hold. Then if $x \in l(p,s)$ we have for each n, assuming $r_n \leq 1$ for all n,

$$|A_n(x)|^{r_n} \leq \left\{ \sum_k |a_{n,k} x_k| \right\}^{r_n}$$

$$= \left\{ \sum_k k^{s/p_k} k^{-s/p_k} |x_k| B^{1/r_n} B^{-1/r_n} |a_{n,k}| \right\}^{r_n}$$

$$\leq \left\{ \sum_k k^{s(q_k - 1)} |a_{n,k}|^{q_k} B^{-q_k/r_n} + \sum_k B^{p_k/r_n} k^{-s} |x_k|^{p_k} \right\}^{r_n}$$

$$\le \{T(B)\}^{r_n} + B^H\{g^H(x)\}^{r_n}$$

$$\le T(B) + 1 + B^H\{g^H(x) + 1\}$$

which implies $A \in (l(p,s), l_\infty(r))$.

Now let $a \in (l(p,s), l_\infty(r))$; then $(a_{n,1}, a_{n,2}, \ldots) \in l(p,s)^+$ for each n and so, by Theorems 3(i) and 4, $A_n \in l(p,s)^*$ for all n. By Theorem 5 there exist M>1 and G≥1 such that $|A_n(x)|^{r_n} \le G$ for all n and all $x \in l(p,s)$ with $g(x) \le 1/M$. Then $\left|\sum_k G^{-1/r_n} a_{n,k} x_k\right| \le 1$ n=1,2,..., if $g(x) \le 1/M$. Write $\Gamma = (G^{-1/r_n} a_{n,k})$, and choose any $x \in l(p,s)$. By the continuity of scaler multiplication on l(p,s) there is a K≥1 such that $g(K^{-1}x) \le 1/M$, whence $\left|\sum_k G^{-1/r_n} a_{n,k} x_k\right| \le K$ for all n. Thus we see that $\Gamma \in (l(p,s), l_\infty)$ and so by Theorem 8(i) there is a D>1 such that

$\sup_n \sum_k k^{s(q_k-1)} D^{-q_k} |G^{-1/r_n} a_{n,k}|^{q_k} < \infty$. Writing B=GD and using the fact that $D^{r_n} \le D$ for all n, we obtain (9).

Theorem 10. (i) Let $0 < \inf_k p_k \le p_k \le 1$ for each $k \in N$, and let r be bounded. Then $A \in (l(p,s), c_0(r))$ if and only if

(11) $\quad |a_{n,k}|^{r_n} \to 0 (n \to \infty)$ for every $k \in N$

and,

(12) $\quad \lim_M \lim\sup_n \left\{\sup_k k^{s/p_k} |a_{n,k}| M^{-1/p_k}\right\}^{r_n} = 0$.

(ii) Let $1 < p_k \le \sup_k p_k = H < \infty$ and $p_k^{-1} + q_k^{-1} = 1$ for each $k \in N$, and let r be bounded. Then $A \in (l(p,s), c_0(r))$ if and only if (11) holds and, for every D≥1,

(13) $\lim_B \lim\sup_n \left\{\sum_k k^{s(q_k-1)} D^{-q_k/r_n} B^{-q_k} |a_{n,k}|^{q_k}\right\}^{r_n}$.

Proof: (i) Lat $A \in (l(p,s), c_0(r))$; since $(l(p,s), c_0(r)) \subseteq (l(p,s), l_\infty(r))$ then as above we have $A_n \in X^*$ and

$\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k} k^{s/p_k}$ whenever $\|A_n\|$ is defined, for each $n \in N$. Then, by Theorem 6 (iii), (12) must hold. also $x = e^{(k)} \in l(p,s)$ (k=1,2,...) we obtain (11).

Conversely if (11) and (12) hold we can show that $A_n \in l(p,s)^*$. With $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k} x^{s/p_k}$ whenever $\|A_n\|$ is defined, for each $n \in N$; also $(e^{(k)})$ is a basis in l(p,s) by Lemma 2. Then by Theorem 6(i) we can deduce that $A \in (l(p,s), c_0(r))$.

(ii) Define A by (10) on l(p,s), for each $n \in N$. First we prove the necessety: let $A \in (l(p,s), c_0(r))$. Obviously we must have (11) and as in Theorem 9 (ii) we see that $A_n \in l(p,s)^*$ for all n. If $A \in (l(p,s), c_0(r))$ then $(D^{1/r_n} a_{n,k}) \in (l(p,s), c_0(q))$ for all D>1, so it is enough to show that (13) holds for D=1. Since $c_0(r) \subseteq l_\infty(r)$ and using Theorem 8 (i) there isa B>1 such that

$T_n \equiv \sum_k k^{s(q_k-1)} B^{-Hq_k} |a_{n,k}|^{q_k} \le 1$ for every $n \in N$. Choose any n, and define $x_k^{(n)} = B^{-Hq_k} \operatorname{sgn} a_{n,k} |a_{n,k}|^{q_k-1} k^{s(q_k-1)}$ for each k; then

$$g^H(x^{(n)}) = \sum_k k^{-s} k^{s(q_k-1)p_k} B^{-Hq_k p_k} |a_{n,k}|^{q_k p_k}$$

$$= \sum_k k^{s(q_k-1)} B^{-Hq_k - Hp_k} |a_{n,k}|^{q_k}$$

$$\leq B^{-H} \sum_k k^{s(q_k-1)} B^{-Hq_k} |a_{n,k}|^{q_k}$$

$$\leq B^{-H}$$

and

$$A_n(x^{(n)}) = \sum_k a_{n,k} x_k^{(n)}$$

$$= \sum_k a_{n,k} k^{s(q_k-1)} B^{-Hq_k} |a_{n,k}|^{q_k-1} \operatorname{sgn} a_{n,k}$$

$$= T_n,$$

whence $\|A_n\|_B \geq T_n$ for each n. By Theorem 6(iii) we must have $\lim_B \limsup_n (\|A_n\|_B)^{r_n} = 0$, whence (13) holds with D=1.

For the sufficiency, let (11) be true and let (13) hold for all $D \geq 1$. It follows that $A_n \in l(p,s)^*$ for all $n \in N$. Since $(e^{(k)})$ is a basis in $l(p,s)$ and using Theorem 6(i) it is enough to show that $\lim_B \limsup_n (\|A_n\|_B)^{r_n} = 0$. Choose $\varepsilon, 0 < \varepsilon \leq 1$, and $D > 2/\varepsilon$. There exist $B > 1$ and m such that

$$\left\{ \sum_k k^{s(q_k-1)} D^{-q_k/r_n} B^{-q_k} |a_{n,k}|^{q_k} \right\}^{r_n} < \varepsilon / 2$$

if $n \geq m$. Then if $g(x) \leq 1/B$ and if $n \geq m$ we have

$$|A_n(x)|^{r_n} \leq \left\{ \sum_k |a_{n,k}| D^{1/r_n} B^{-1} B D^{-1/r_n} k^{s/p_k} k^{-s/p_k} |x_k| \right\}^{r_n}$$

$$\leq \left\{ \sum_k (|a_{n,k}|^{q_k} D^{q_k/r_n} B^{-q_k} k^{s(q_k-1)} + D^{-p_k/r_n} B^{p_k} k^{-s} |x_k|^{p_k}) \right\}^{r_n}$$

$$\leq \left\{ \sum_k (|a_{n,k}|^{q_k} D^{q_k/r_n} B^{-q_k} k^{s(q_k-1)} \right\}^{r_n} + \left\{ D^{-1/r_n} B^H g^H(x) \right\}^{r_n}$$

$$\leq \varepsilon / 2 + \left\{ D^{-1/r_n} B^H g^H(x) \right\}^{r_n} < \varepsilon,$$

and this completes the proof.

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