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THE STATE POLYNOMIAL OF KNOT $K_{(3,3)}$

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Abstract : In this work, the state polynomial of knot $K_{(3,3)}$, which is called 7_4, is calculated. From this polynomial, the Alexander-Conway polynomial $1+4z^2$ of knot $K_{(3,3)}$ is obtained.

Key Words : Knot, state polynomial, Alexander-Conway polynomial

1. Introduction

Knots and links in three-dimensional space may be understood through their planer projections. A knot is usually drawn as a schematic snapshot, with crossing indirected by broken line segments. The Figure 1 represents the knot $K_{(3,3)}$. We shall refer to such a picture as a knot diagram. The projection corresponding to such a diagram forms a (directed) multi-graph in the plane, with four edges incident to each vertex. A (directed) planer graph with four edges incident to each vertex will be termed a **universe**. The universe of knot $K_{(3,3)}$ is illustrated in the Figure 2. These universe have **singularities** (the crossing); they will also **states, black holes, white holes** end **stars.**

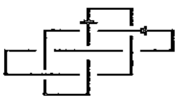


Figure 1

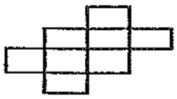


Figure 2

A state of a universe is an assignment of one marker Per vertex as in the Figure 3.

Figure 3

Such each region in the graph receives no more than one marker. The state of universe of knot $K_{(3,3)}$ is illustrated in the Figure 4.

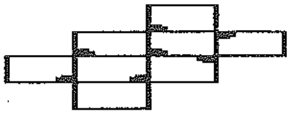


Figure 4

Two regions of the state will be free of markers (since the number of regions exceeds the number of vertices by two in a connected universe). These free regions are inhabited by the **stars**(*). [1].

It should be mentioned at once that each universe has states. In fact, states **with stars in adlacent are in one-to-one correspondence with Jordan trails on the universe.** A **Jordan trail** is an (unoriented) path that traverses every edge of the universe once end forms a simple closed curve in the plane. This correspondence is obtained by regarding each state marker as an instruction to split its crossing according to the Figure 5.

Figure 5

By splitting all crossing in a state, the Jordan trail automatically appears. Conversely, a choice of stars at Jordan trail determines a specific state. The process is illustrated in the Figure 6 for the knot $K_{(3,3)}$.

Figure 6

This correspondence underlines the importance of states with adlacent stars; further reference to states will assume star adjacency unless otherwise specified. The state markers are classified into the categories **black holes**, **white holes**, up and **down** according to placement with respect to the crossing orientation (see Figure 7).

Figure 7

The sign of a state S, $\sigma(S)$, is defined by the formula $\sigma(S)=(-1)^b$ where b denotes the number of black holes in S. Just as the sign of a permutation changes under single transpositions of its elements, so does the sign of a state change under state **transposition.** A state transposition is a movement from one state to another that is obtained by switching a pair of state markers as indicated the Figure 8.

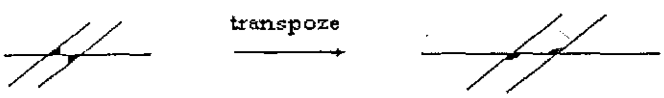


Figure 8

Note that in a state transposition both states markers rotate by one quarter turn in the same clock-direction. A state transposition in which the markers turn clockwise(counter-clockwise) will be termed a clockwise(counter-clockwise) movement. A state is said to be **clocked** if it admits only clockwise movement, **mixed** if it admits only counter-clockwise movements and **counter-clockwise** only counter-clockwise movements.

Patterns of black and white holes in the states give a rise to a **Duality Conjecture** and to the series of results bridging combinatorial and the topology of knots and links [1]. ,

Duality Conjecture : Let δ be the collection of states of an oriented universe U with a choice of fixed adjacent stars. Let $N(r, s, \delta)=N(r, s)$ denote the number of states in δ with r black holes and s white holes.

Theorem : $N(r, s)$ is independent of star placement. That is, if δ is another state collection arising from a different choice of fixed stars, then $N(r, s, \delta) = N(r, s, \delta')$ for all r and s [1]. This independence result depends on crucially and subtly on the Clock Theorem [1].

It is verified by interpreting the state **polynomial** (belonging to the polynomial ring in variables B and W over the integers $Z[B, W]$)

$$F(S) = \sum_{r,s} (-1)^r N(r, s, \delta) B^r W^s$$

as a determinant of a matrix associated with the universe and with δ. The signs of the permutations that occur in the expansion of this determinant coincide with the signs $(-1)^r$, and these are the signs of the states being enumerated.

A generalisation of the state polynomial marks the transition into the theory of knots. A knot-diagram is a universe with extra structure at the crossings. To create a knot-diagram from a given universe entails a two-fold choice at each crossing.

Hence 2^c knot-diagrams project to a universe with c crossing. It is convenient to designate these choices by placing a **code** at each crossing. Our codes take the forms in the Figure 9.

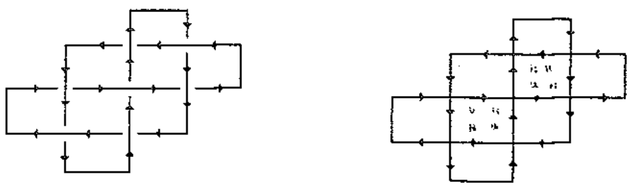


Figure 9

Thus a knot or link diagram is an oriented universe with standard or reverse codes at each crossing.

In standard code the labels B and W hover over the potential black and white holes respectively. The labels are flipped in the reverse code. The knot or link obtained by labelling a universe entirely with standard (reverse) code will be called a **standard (reverse) knot.** The reverse knot is the mirror image of the corresponding a standard knot.

Let K be a knot and S a state, both shrink the same underlying universe U. We define an inner-product $\langle K/S \rangle \in Z[B,W]$ and a state polynomial $\langle K/\delta \rangle$,

$$\langle K/\delta \rangle = \sum_{S\in\delta} \langle K/S \rangle$$

so that when K is standard this new polynomial coincides with the original state polynomial of δ. In order to do this the inner product is defined as follows:

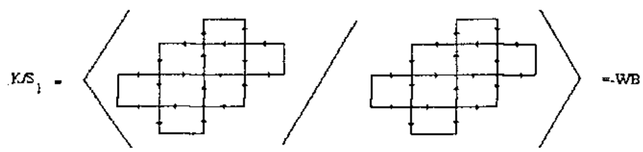
Superimpose K and S on the universe U. Let x denote the number of coincides of B- labels with state markers. Then,

$$\langle K/S \rangle = \sigma(S) W^x B^y$$

When K is standard, x is the number of white holes and y is the number of black holes in S.

2.The State Polynomial of Knot $K_{(3,3)}$

Let K be the knot $K_{(3,3)}$. Then, K has fifteen states and one of them is



and consideration of fourteen other knot states $K_{(3,3)}$ show that

$$\langle K/\delta \rangle = \sum_{i=1}^{15} \langle K/S_i \rangle = 4B^2+4W^2-7WB = 4(B-W)^2+WB$$

Now let us show that how it is done: The states a_1, a_2, a_3, a_4 ; b_1, b_2, b_3, b_4 ; c_1, c_3, c_4 (it is not allowed for c_2) and d_1, d_2, d_3, d_4 are written in the figure 10.

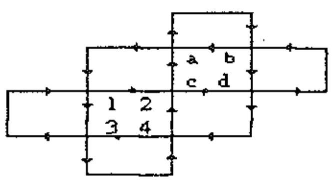


Figure 10

Let us set

$S_1=a_1, S_2=a_2, S_3=a_3, S_4=a_4$
$S_5=b_1, S_6=b_2, S_7=b_3, S_8=b_4$
$S_9=c_1, S_{10}=c_3, S_{11}=c_4$, (it is not allowed for c_2)
$S_{12}=d_1, S_{13}=d_2, S_{14}=d_3, S_{15}=d_4.$

Hence, these states are ordered as follows:

$(x=1, y=1);\ \sigma(a_1)=-1$ and $\langle K/a_1\rangle=-WB$
$(x=0, y=2);\ \sigma(a_2)=1$ and $\langle K/a_2\rangle=B^2$
$(x=0, y=2);\ \sigma(a_3)=1$ and $\langle K/a_3\rangle=B^2$
$(x=1, y=1);\ \sigma(a_4)=-1$ and $\langle K/a_4\rangle=-WB$
$(x=2, y=0);\ \sigma(b_1)=1$ and $\langle K/b_1\rangle=W^2$
$(x=1, y=1);\ \sigma(b_2)=-1$ and $\langle K/b_2\rangle=-WB$
$(x=1, y=1);\ \sigma(b_3)=-1$ and $\langle K/b_3\rangle=-WB$
$(x=2, y=0);\ \sigma(b_4)=1$ and $\langle K/b_4\rangle=W^2$
$(x=2, y=0);\ \sigma(c_1)=1$ and $\langle K/c_1\rangle=W^2$
$(x=1, y=1);\ \sigma(c_3)=-1$ and $\langle K/c_3\rangle=-WB$
$(x=2, y=0);\ \sigma(c_4)=1$ and $\langle K/c_4\rangle=W^2$
$(x=1, y=1);\ \sigma(d_1)=-1$ and $\langle K/d_1\rangle=-WB$
$(x=0, y=2);\ \sigma(d_2)=1$ and $\langle K/d_2\rangle=B^2$
$(x=0, y=2);\ \sigma(d_3)=1$ and $\langle K/d_3\rangle=B^2$
$(x=1, y=1);\ \sigma(d_4)=-1$ and $\langle K/d\ \rangle=-WB$

Therefore,

$$\langle K/\delta\rangle=\sum_{i=1}^{15}\langle K/S_i\rangle=4B^2+4W^2-7WB=4(B-W)^2+WB$$

Simply set $WB=1$ and let $z=B-W$, then $\langle K/\delta\rangle$ becomes a polynomial in z, $\nabla_K(z)$.Where $\nabla_K(z)$ is a topological invariant of K. In the knot $K_{(3,3)}$ we have

$$\nabla_K(z) = 1 + 4z^2.$$

The polynomial $\nabla_K(z)$ is identical what I called the Alexander-Conway polynomial of knot $K_{(3.3)}$ ([2],[3],[4],[5]).

References

[1]. L.H. Kauffman. Formal Knot Theory Mathematical Notes 30. Princeton University Press,1983.

[2]. L.H. Kauffman. The Conway Polynomial. Topology 20 (1980), 101-108.

[3]. J.W. Alexander Topological invariant of Knots and Links. Trans. Am. Math. Soc. 30 (1923),. 275-306.

[4]. James M. Van Buskirk. Positive Knots Have Positive Conway Polynomials. Knot Theory and Manifolds. University of Oregon, Eugone New York, 1985.

[5]. J.H. Conway. An Enumaration of Knot and Links and Some of Their Algebraic Properties, Computationals Problems in Abstract Algebra, Pergamon Press, New York (1970), 329-358.

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