Existence of Positive Solutions for a Discrete Boundary Value Problem

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Abstract. In this paper we study the existence and nonexistence of positive solutions for a class of nonlinear difference systems subject to some $m+1$–point boundary conditions. The arguments for existence of solutions are based upon the Schauder fixed point theorem.

Keywords: Difference equations, multi-point boundary value problem, positive solution, fixed point theorem.

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1. Introduction

We consider the discrete system with second-order differences

$$(S) \qquad \begin{cases} \Delta^2 u_{n-1} + b_n f(v_n) = 0, & n = \overline{1, N-1} \\ \Delta^2 v_{n-1} + c_n g(u_n) = 0, & n = \overline{1, N-1}, \quad (N \geq 2), \end{cases}$$

with $m+1$-point boundary conditions

$$(BC) \qquad \begin{cases} \beta u_0 - \gamma \Delta u_0 = 0, \quad u_N - \sum_{i=1}^{m-2} a_i u_{\xi_i} = b, \\ \beta v_0 - \gamma \Delta v_0 = 0, \quad v_N - \sum_{i=1}^{m-2} a_i v_{\xi_i} = b, \quad m \geq 3, \end{cases}$$

where Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$ and $b > 0$.

The above problem is equivalent to

$$\begin{cases} u_{n+1} - 2u_n + u_{n-1} + b_n f(v_n) = 0 \\ v_{n+1} - 2v_n + v_{n-1} + c_n g(u_n) = 0, \quad n = \overline{1, N-1}, \end{cases}$$

with the conditions

$$\begin{cases} (\beta + \gamma) u_0 = \gamma u_1, \quad u_N - \sum_{i=1}^{m-2} a_i u_{\xi_i} = b \\ (\beta + \gamma) v_0 = \gamma v_1, \quad v_N - \sum_{i=1}^{m-2} a_i v_{\xi_i} = b. \end{cases}$$

In this paper we shall investigate the existence and nonexistence of positive solutions of (S), (BC). In the case $b = 0$ and $b_n = \lambda \widetilde{b}_n$, $c_n = \mu \widetilde{c}_n$, λ, $\mu > 0$, the existence of positive solutions with respect to a cone has been studied in [11]. In [10] the authors studied the

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existence and nonexistence of positive solutions for the m-point boundary value problem on time scales

$$\begin{cases} u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \ \ t \in (0,T) \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \ \ u(T) - \sum_{i=1}^{m-2} a_i u(\xi_i) = b, \ \ m \geq 3, \ \ b > 0. \end{cases}$$

In recent years the existence of positive solutions of multi-point boundary value problems for second-order or higher-order differential or difference equations has been the subject of investigations by many authors (see also [1]–[9], [12]–[15]).

We shall suppose that the following conditions are verified

(A1) $b_n, \, c_n \geq 0$ for $n = \overline{1, N-1}$; $\beta, \, \gamma \geq 0, \, \beta + \gamma > 0$; $a_i > 0, \, i = \overline{1, m-2}$, $a_{m-2} \geq 1$; $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < N$; $b > 0$, $N > \sum_{i=1}^{m-2} a_i \xi_i$, $d = \beta \left(N - \sum_{i=1}^{m-2} a_i \xi_i \right) + \gamma \left(1 - \sum_{i=1}^{m-2} a_i \right) > 0$.

(A2) There exist $n_0, \, \widetilde{n}_0 \in \{\xi_{m-2}, ..., N\}$ such that $b_{n_0} > 0, \ c_{\widetilde{n}_0} > 0$.

(A3) $f, \, g : [0, \infty) \to [0, \infty)$ are continuous functions that satisfy the conditions

a) There exists $c > 0$ such that $f(u) < \dfrac{c}{L}, \ \ g(u) < \dfrac{c}{L}$, for all $u \in [0, c]$;

b) $\lim_{u \to \infty} \dfrac{f(u)}{u} = \infty, \ \ \lim_{u \to \infty} \dfrac{g(u)}{u} = \infty$,

where $L = \max \left\{ \dfrac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i, \ \dfrac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) c_i \right\}$.

2. Preliminaries

In this section we shall present some auxiliary results from [10] and [11], related to the following second-order difference system with boundary conditions

$$\Delta^2 u_{n-1} + y_n = 0, \ \ n = \overline{1, N-1} \tag{1}$$

$$\beta u_0 - \gamma \Delta u_0 = 0, \ \ u_N - \sum_{i=1}^{m-2} a_i u_{\xi_i} = 0. \tag{2}$$

Lemma 2.1. *([10], [11]) If $\beta + \gamma \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < N$ and $d \neq 0$, then the solution of (1), (2) is given by*

$$\begin{aligned} u_n = \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i) y_i - \frac{n\beta + \gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i - 1} (\xi_i - j) y_j \\ - \sum_{i=1}^{n-1} (n-i) y_i, \ \ i = \overline{0, N}. \end{aligned} \tag{3}$$

We use the conventions $\sum_{i=1}^{0} z_i = 0$ and $\sum_{i=1}^{-1} z_i = 0$.

Lemma 2.2. *([11]) Under the assumptions of 2.1, the Green function for the boundary value problem (1), (2) is given by*

$$G(n,i) = \begin{cases} \dfrac{n\beta+\gamma}{d}(N-i) - \dfrac{n\beta+\gamma}{d}\sum_{k=1}^{m-2} a_k(\xi_k - i) - (n-i), \\ \qquad if \;\; i < \xi_1, \; n \geq i, \\ \qquad (for \; n=0 \; or \; n=1 \; without \; term \; (n-i)), \\ \dfrac{n\beta+\gamma}{d}(N-i) - \dfrac{n\beta+\gamma}{d}\sum_{k=1}^{m-2} a_k(\xi_k - i), \;\; if \;\; n \leqslant i < \xi_1, \\ \dfrac{n\beta+\gamma}{d}(N-i) - \dfrac{n\beta+\gamma}{d}\sum_{k=j}^{m-2} a_k(\xi_k - i) - (n-i), \\ \qquad if \;\; \xi_{j-1} \leqslant i < \xi_j, \; n \geq i, \;\; j = \overline{2, m-2}, \\ \dfrac{n\beta+\gamma}{d}(N-i) - \dfrac{n\beta+\gamma}{d}\sum_{k=j}^{m-2} a_k(\xi_k - i), \\ \qquad if \;\; \xi_{j-1} \leqslant i < \xi_j, \; n \leqslant i, \;\; j = \overline{2, m-2}, \\ \dfrac{n\beta+\gamma}{d}(N-i) - (n-i), \;\; if \; \xi_{m-2} \leqslant i \leqslant n, \\ \dfrac{n\beta+\gamma}{d}(N-i), \;\; if \; i \geq \xi_{m-2}, \; n \leqslant i, \end{cases}$$

and we have $u_n = \sum_{i=1}^{N-1} G(n,i) y_i, \;\; n = \overline{0,N}$.

Lemma 2.3. *([10], [11]) If $d > 0$, $\beta, \gamma \geq 0$, $\beta + \gamma > 0$, $a_i > 0$ for all $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < N$, $\sum_{i=1}^{m-2} a_i \xi_i \leqslant N$ and $y_n \geq 0$, for all $n = \overline{1, N-1}$, then the solution u_n, $n = \overline{0,N}$ of problem (1), (2) satisfies $u_n \geq 0$, for all $n = \overline{0,N}$.*

Lemma 2.4. *([11]) If $d > 0$, $\beta, \gamma \geq 0$, $\beta + \gamma > 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < N$, $a_i > 0$, $i = \overline{1, m-2}$, $a_{m-2} \geq 1$, $N \geq \sum_{i=1}^{m-2} a_i \xi_i$, $y_n \geq 0$ for all $n = \overline{1, N-1}$, then the solution of problem (1), (2) satisfies*

$$u_n \leqslant \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) y_i, \;\; \forall\, n = \overline{0,N},$$
$$u_{\xi_j} \geq \frac{\beta \xi_j + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) y_i, \;\; \forall\, j = \overline{1, m-2}.$$

Lemma 2.5. *([10]) We assume that* β, $\gamma \geq 0$, $\beta+\gamma > 0$, $d > 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < N$, $a_i > 0$ *for all* $i = \overline{1, m-2}$, $N > \sum_{i=1}^{m-2} a_i\xi_i$ *and* $y_n \geq 0$ *for all* $n = \overline{1, N-1}$. *Then the unique solution of problem (1), (2) verifies the relation* $\inf_{n=\overline{\xi_1,N}} u_n \geq r\|u\|$, *where*

$$r = \min_{2\leqslant s\leqslant m-2} \left\{ \frac{\xi_1}{N}, \frac{\sum_{i=1}^{m-2} a_i(N-\xi_i)}{N - \sum_{i=1}^{m-2} a_i\xi_i}, \frac{\sum_{i=1}^{m-2} a_i\xi_i}{N}, \frac{\sum_{i=1}^{s-1} a_i\xi_i + \sum_{i=s}^{m-2} a_i(N-\xi_i)}{N - \sum_{i=s}^{m-2} a_i\xi_i} \right\}$$

and $\|u\| = \sup_{n=\overline{0,N}} |u_n|$.

3. Main results

We shall firstly present an existence result for the positive solutions of (S), (BC).

Theorem 3.1. *Assume that the assumptions (A1), (A2), (A3)a hold. Then the problem* (S), (BC) *has at least one positive solution for* $b > 0$ *sufficiently small.*

Proof. We consider the problem

$$\begin{cases} \Delta^2 h_n = 0 \\ \beta h_0 - \gamma \Delta h_0 = 0, \quad h_N = \sum_{i=1}^{m-2} a_i h_{\xi_i} + 1. \end{cases} \tag{4}$$

The solution $(h_n)_{n=\overline{2,N}}$ of $(4)_1$ is given by $h_n = nh_1 - (n-1)h_0$, $n = \overline{2,N}$. Because $\beta h_0 - \gamma(h_1 - h_0) = 0$, that is $h_0 = \frac{\gamma}{\beta+\gamma}h_1$, we get $h_n = \frac{n\beta+\gamma}{\beta+\gamma}h_1$, $n = \overline{2,N}$. By the condition $h_N = \sum_{i=1}^{m-2} a_i h_{\xi_i} + 1$ we obtain $\frac{N\beta+\gamma}{\beta+\gamma}h_1 = \sum_{i=1}^{m-2} a_i \frac{\beta\xi_i+\gamma}{\beta+\gamma}h_1 + 1$, which implies $h_1 = \frac{\beta+\gamma}{d}$. So $h_n = \frac{n\beta+\gamma}{d}$, $n = \overline{2,N}$.

Therefore the solution of (4) is

$$h_n = \frac{n\beta+\gamma}{d}, \quad n = \overline{0,N}. \tag{5}$$

We now define $(x_n)_{n=\overline{0,N}}$, $(y_n)_{n=\overline{0,N}}$ by

$$\begin{cases} u_n = x_n + bh_n \\ v_n = y_n + bh_n, \quad n = \overline{0,N}. \end{cases}$$

Then (S), (BC) can be equivalently written as

$$\begin{cases} \Delta^2 x_{n-1} + b_n f(y_n + bh_n) = 0, \quad n = \overline{1,N-1} \\ \Delta^2 y_{n-1} + c_n g(x_n + bh_n) = 0, \quad n = \overline{1,N-1}, \end{cases} \tag{6}$$

with the boundary conditions

$$\begin{cases} \beta x_0 - \gamma \Delta x_0 = 0, \;\; x_N = \displaystyle\sum_{i=1}^{m-2} a_i x_{\xi_i} \\ \beta y_0 - \gamma \Delta y_0 = 0, \;\; y_N = \displaystyle\sum_{i=1}^{m-2} a_i y_{\xi_i}. \end{cases} \tag{7}$$

Using the Green function given in 2.2, a pair $((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}})$ is a solution of problem (6), (7) if and only if

$$\begin{cases} x_n = \displaystyle\sum_{i=1}^{N-1} G(n,i) b_i f\left(\sum_{j=1}^{N-1} G(i,j) c_j g(x_j + bh_j) + bh_i\right), \;\; n = \overline{0,N}, \\ y_n = \displaystyle\sum_{i=1}^{N-1} G(n,i) c_i g(x_i + bh_i), \;\; n = \overline{0,N}, \end{cases} \tag{8}$$

where $(h_n)_n$ is given by (5).

We consider the Banach space $X = \mathbb{R}^{N+1}$ with supremum norm $\|\cdot\|$ and we define the set $K = \{(x_n)_{n=\overline{0,N}}, \; 0 \leqslant x_n \leqslant c, \;\; \forall\, n = \overline{0,N}\} \subset X$.

We also define the operator $\Lambda : K \to X$ by

$$\Lambda(x) = \left(\sum_{i=1}^{N-1} G(n,i) b_i f\left(\sum_{j=1}^{N-1} G(i,j) c_j g(x_j + bh_j) + bh_i\right)\right)_{n=\overline{0,N}}, \;\; x = (x_n)_{n=\overline{0,N}} \in K.$$

For sufficiently small $b > 0$, by (A3)a we deduce

$$f(y_n + bh_n) \leqslant \frac{c}{L}, \;\; g(x_n + bh_n) \leqslant \frac{c}{L}, \;\; \forall\,(x_n)_n, \, (y_n)_n \in K.$$

Then for any $x = (x_n)_n \in K$ we have, using 2.3, that $(\Lambda x)_n \geq 0, \;\forall\, n \in \overline{0,N}$. By 2.4 we also have

$$\begin{aligned} y_j &\leqslant \frac{\beta N + \gamma}{d} \sum_{k=1}^{N-1} (N-k) c_k g(x_k + bh_k) \leqslant \frac{c}{L} \cdot \frac{\beta N + \gamma}{d} \sum_{k=1}^{N-1} (N-k) c_k \\ &\leqslant \frac{c}{L} \cdot L = c, \;\; \forall\, j = \overline{1, N-1} \end{aligned}$$

and

$$\begin{aligned} \Lambda(x)_n &\leqslant \frac{\beta N + \gamma}{d} \sum_{k=1}^{N-1} (N-k) b_k f(y_k + bh_k) \leqslant \frac{c}{L} \cdot \frac{\beta N + \gamma}{d} \sum_{k=1}^{N-1} (N-k) b_k \\ &\leqslant \frac{c}{L} \cdot L = c, \;\; \forall\, n = \overline{0,N}. \end{aligned}$$

Therefore $\Lambda(K) \subset K$.

Using standard arguments we deduce that Λ is completely continuous (Λ is compact because for any bounded set $B \subset K$, $\Lambda(B) \subset K$ is bounded, so in $\mathbb{R}^{N+1}$ is relatively compact, and Λ is continuous because f, g are continuous). By the Schauder fixed point theorem, we conclude that Λ has a fixed point $(x_n)_{n=\overline{0,N}} \in K$. This element together with $(y_n)_{n=\overline{0,N}}$ given by (8) represent a solution for (6), (7). This shows that our problem

(S), (BC) has a positive solution $u_n = x_n + bh_n,\ v_n = y_n + bh_n,\ n = \overline{0, N}$ for sufficiently small $b > 0$. $\square$

In the following theorem we shall present sufficient conditions for nonexistence of positive solutions of (S), (BC).

Theorem 3.2. *Assume that the assumptions (A1), (A2), (A3)b hold. Then the problem (S), (BC) has no positive solution for b sufficiently large.*

Proof. We shall follows the same steps as those used in the proof of Theorem 3.2 from [10]. We suppose that $(u_n)_n$ is a positive solution of (S), (BC). Then $x_n = u_n - bh_n,\ y_n = v_n - bh_n,\ n = \overline{0, N}$ is solution for (6), (7), where $(h_n)_{n=\overline{0,N}}$ is the solution of problem (4). By Lemma 2.3 we have $x_n \geq 0,\ y_n \geq 0$, for all $n = \overline{0, N}$, and by (A2) we deduce that $\|x\| > 0,\ \|y\| > 0$. Using Lemma 2.5 we also have $\inf_{n=\overline{\xi_1,N}} x_n \geq r\|x\|$ and $\inf_{n=\overline{\xi_1,N}} y_n \geq r\|y\|$, where r is defined in Lemma 2.5.

Using now (5) - the expression for $(h_n)_{n=\overline{0,N}}$ we deduce that

$$\inf_{n=\overline{\xi_1,N}} h_n = \frac{\beta\xi_1 + \gamma}{d} \geq \frac{\xi_1 h_N}{N} = \frac{\xi_1}{N} \cdot \frac{\beta N + \gamma}{d}.$$

So $\inf_{n=\overline{\xi_1,N}} h_n \geq \dfrac{\xi_1}{N}\|h\|$, $(\|h\| = h_N)$. Then

$$\inf_{n=\overline{\xi_1,N}}(x_n + bh_n) \geq r(\|x\| + b\|h\|) \geq r\|x + bh\|$$

and

$$\inf_{n=\overline{\xi_1,N}}(y_n + bh_n) \geq r(\|y\| + b\|h\|) \geq r\|y + bh\|.$$

We now consider

$$R = \frac{d}{r(\beta\xi_{m-2} + \gamma)} \left(\min \left\{ \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j,\ \sum_{j=\xi_{m-2}}^{N-1} (N-j)b_j \right\} \right)^{-1} > 0.$$

By (A3)b, that is $\lim\limits_{u\to\infty} \dfrac{f(u)}{u} = \infty,\ \lim\limits_{u\to\infty} \dfrac{g(u)}{u} = \infty$, for R defined above we deduce that there exists $M > 0$ such that $f(u) > 2Ru,\ g(u) > 2Ru$, for all $u \geq M$.

We consider $b > 0$ sufficiently large such that

$$\inf_{n=\overline{\xi_1,N}}(x_n + bh_n) \geq M \quad \text{and} \quad \inf_{n=\overline{\xi_1,N}}(y_n + bh_n) \geq M.$$

Then we have

$$y_{\xi_{m-2}} \geq \sum_{j=\xi_{m-2}}^{N-1} \frac{\beta\xi_{m-2} + \gamma}{d}(N-j)c_j g(x_j + bh_j)$$

$$\geq \sum_{j=\xi_{m-2}}^{N-1} \frac{\beta\xi_{m-2} + \gamma}{d}(N-j)c_j \cdot 2R(x_j + bh_j)$$

$$\geq \sum_{j=\xi_{m-2}}^{N-1} \frac{\beta\xi_{m-2}+\gamma}{d}(N-j)c_j \cdot 2R \inf_{k=\overline{\xi_{m-2},N}}(x_k+bh_k)$$
$$\geq \sum_{j=\xi_{m-2}}^{N-1} \frac{\beta\xi_{m-2}+\gamma}{d}(N-j)c_j \cdot 2R \inf_{k=\overline{\xi_1,N}}(x_k+bh_k)$$
$$\geq \sum_{j=\xi_{m-2}}^{N-1} \frac{\beta\xi_{m-2}+\gamma}{d}(N-j)c_j \cdot 2R\delta\|x+bh\|$$
$$\geq 2\|x+bh\| \geq 2\|x\|.$$

And then we obtain

$$\|x\| \leqslant \frac{1}{2}y_{\xi_{m-2}} \leqslant \frac{1}{2}\|y\|. \tag{9}$$

In a similar manner we deduce $x_{\xi_{m-2}} \geq 2\|y+bh\| \geq 2\|y\|$ and so

$$\|y\| \leqslant \frac{1}{2}x_{\xi_{m-2}} \leqslant \frac{1}{2}\|x\|. \tag{10}$$

By (9) and (10) we obtain $\|x\| \leqslant \frac{1}{2}\|y\| \leqslant \frac{1}{4}\|x\|$, which is a contradiction, because $\|x\| > 0$.

Then, when b is sufficiently large, our problem (S), (BC) has no positive solution.

□

4. An example

We consider $b_n = \dfrac{b_0}{N-n}$, $c_n = \dfrac{c_0}{N-n}$ for all $n = \overline{1, N-1}$, $b_0, c_0 > 0$, $\beta = 2$, $\gamma = \frac{1}{4}$, $m = 5$, $\xi_1 = \frac{N}{4}$, $\xi_2 = \frac{N}{2}$, $\xi_3 = \frac{3N}{4}$ $(N = 4M,\ M \geq 1)$, $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{3}$, $a_3 = 1$. Then $d = \frac{2N-7}{48} > 0$ and the assumption $N > \sum_{i=1}^{3} a_i\xi_i$ is verified $(N > \frac{47}{48}N)$.

We consider $f, g : [0,\infty) \to [0,\infty)$, $f(x) = \dfrac{\widetilde{a}x^3}{x+1}$, $g(x) = \dfrac{\widetilde{b}x^3}{x+1}$, with $\widetilde{a}, \widetilde{b} > 0$. We have $\lim_{x\to\infty} \dfrac{f(x)}{x} = \lim_{x\to\infty} \dfrac{g(x)}{x} = \infty$.

Next the constant L from (A3) is in this case

$$L = \max\left\{ \frac{\beta N+\gamma}{d}\sum_{i=1}^{N-1}(N-i)b_i, \frac{\beta N+\gamma}{d}\sum_{i=1}^{N-1}(N-i)c_i \right\}$$
$$= \frac{12(8N+1)(N-1)}{2N-7}\max\{b_0, c_0\}.$$

We choose $c = 1$ and if we select $\widetilde{a}$ and $\widetilde{b}$ satisfying the conditions

$$\widetilde{a} < \frac{2}{L} = \frac{2(2N-7)}{12(8N+1)(N-1)} \min\left\{\frac{1}{b_0}, \frac{1}{c_0}\right\},$$
$$\widetilde{b} < \frac{2}{L} = \frac{2(2N-7)}{12(8N+1)(N-1)} \min\left\{\frac{1}{b_0}, \frac{1}{c_0}\right\},$$

then we obtain $f(x) \leqslant \frac{\widetilde{a}}{2} < \frac{1}{L}$, $g(x) \leqslant \frac{\widetilde{b}}{2} < \frac{1}{L}$, for all $x \in [0,1]$.

Thus all the assumptions (A1)-(A3) are verified. By Theorem 3.1 and Theorem 3.2 we deduce that the nonlinear discrete system

$$\begin{cases} \Delta^2 u_{n-1} + \dfrac{b_0}{N-n} \cdot \dfrac{\widetilde{a} v_n^3}{v_n + 1} = 0 \\ \Delta^2 v_{n-1} + \dfrac{c_0}{N-n} \cdot \dfrac{b u_n^3}{u_n + 1} = 0, \quad n = \overline{1, N-1} \ (N = 4M, \ N \geq 4), \end{cases}$$

with the boundary conditions

$$\begin{cases} u_0 = \frac{u_1}{9}, \ \ u_N - \frac{1}{4} u_{N/4} - \frac{1}{3} u_{N/2} - u_{3N/4} = b, \\ v_0 = \frac{v_1}{9}, \ \ v_N - \frac{1}{4} v_{N/4} - \frac{1}{3} v_{N/2} - v_{3N/4} = b \end{cases}$$

has at least one positive solution for sufficiently small $b > 0$ and no solution for sufficiently large b.

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