Amsterdam Properties of Complete Quasi-metric Spaces

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Abstract. In this paper, we establish relationships between complete quasi-metric spaces and some Amsterdam properties. It is shown that every left K-sequentially complete quasi-pseudo-metric space is weakly hypocompact; and every regular quasi-developable left K-sequentially complete quasi-metric space is hypocompact. Examples of quasi-metric spaces which satisfy some Amsterdam properties but admit no any compatible left K-sequentially complete quasi-metric are provided.

Keywords: Baire space, base-compact, left K-sequentially complete, hypocompact, quasi-pseudo-metric, weakly hypocompact.

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1. INTRODUCTION

The Baire category theorem in mathematics says that if a topological space X is either completely metrizable or locally compact Hausdorff, then the intersection of countably many dense open sets of X is still dense. Spaces which have the topological property described in the conclusion of the Baire category theorem, i.e., spaces in which the intersection of any countable collection of dense open sets is dense, are called *Baire spaces*. Since this theorem has numerous applications in analysis and topology, it is interesting to consider the following classical problem.

Problem 1.1. *Is there a natural class of spaces which contains all completely metrizable spaces and all locally compact Hausdorff spaces such that the conclusion of the Baire category theorem remains valid?*

Čech [4] gave one solution to this problem by introducing the class of Čech complete spaces, where he proved that every Čech complete space is Baire, and a metrizable space is completely metrizable if and only if it is Čech complete. Recall that a Tychonoff space X is *Čech complete* if it is a G_δ-set of its Stone-Čech compactification βX [6]. In studying

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Problem 1.1, de Groot and his students introduced various completeness properties which were called *Amsterdam properties*, see [1]. In particular, de Groot [9] proved that a metrizable space is completely metrizable if and only if it is subcompact, where a regular space X is called *subcompact* if there is a base $\mathfrak{B}$ such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathfrak{B}$ is a regular filterbase. Recall that a collection $\mathcal{F}$ of nonempty subsets of X is called a *regular filterbase* if for every pair of sets $F_1, F_2 \in \mathcal{F}$ there is an $F_3 \in \mathcal{F}$ such that $\overline{F_3} \subseteq F_1 \cap F_2$. Subcompactness is just one of the Amsterdam properties, in what follows we shall mention a few more of them. A space X is called *base-compact* [1] (resp. *hypocompact* [5]) if there is a base $\mathfrak{B}$ such that $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$ for any centered system (resp. filterbase) $\mathcal{F} \subseteq \mathfrak{B}$. Furthermore, a space is called *weakly base-compact* (resp. *weakly hypocompact* [5]) if there is a sequence of bases $\{\mathfrak{B}_n : n \in \mathbb{N}\}$ such that $\{B_n : n \in \mathbb{N}\}$ is a centered system (resp. filterbase) and $B_n \in \mathfrak{B}_{k_n}$ with $k_1 < k_2 < \cdots$ imply $\bigcap_{n \in \mathbb{N}} \overline{B_n} \neq \emptyset$. Countably base-compact (subcompact, hypocompact) spaces can be defined similarly.

Since every completely metrizable space has all the previously mentioned Amsterdam properties, we consider the following natural problem.

Problem 1.2. *Does a "complete" quasi-metric space X have any of the previously mentioned Amsterdam properties?*

Recall that a (resp. *non-Archimedean*) *quasi-pseudo-metric* d on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that (i) $d(x, x) = 0$ for all $x \in X$; (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$ (resp. $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$) for all $x, y, z \in X$. The pair (X, d) is called a (resp. *non-Archimedean*) *quasi-pseudo-metric* space. If d satisfies the additional condition that $d(x, y) = 0$ implies $x = y$ then d is called a (resp. *non-Archimedean*) *quasi-metric*. The set $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is the *d-ball* with centre x and radius ε. A topological space (X, τ) is called (resp. *non-Archimedeanly*) *quasi-pseudo-metrizable* if there is a (resp. non-Archimedean) quasi-pseudo-metric d on X such that $\tau = \tau_d$.

In the literature, there are several different versions of completeness for quasi-metric spaces. In this paper, we mainly consider two of them, namely left K-sequential completeness and left p-sequential completeness. Let (X, d) be a quasi-pseudo-metric space. A sequence $\{x_n : n \in \mathbb{N}\}$ in (X, d) is called *left K-Cauchy* (resp. *weakly left K-Cauchy*) if for each $\varepsilon > 0$ there is some $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $k \leqslant n \leqslant m$ (resp. $d(x_k, x_m) < \varepsilon$ for all $m \geq k$); and (X, d) is said to be *left K-sequentially complete* (resp. *weakly left K-sequentially complete*) if every left K-Cauchy (resp. weakly left K-Cauchy)

sequence converges to some point $x \in X$. Further, a sequence $\{x_n : n \in \mathbb{N}\}$ in (X, d) is said to be *left p-Cauchy* if for every $\varepsilon > 0$ there are $x \in X$ and $k \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n > k$; and (X, d) is said to be *left p-sequentially complete* if every left p-Cauchy sequence converges to some point $x \in X$. A good reference for these notions is [13].

Note that Problem 1.2 is also relevant to another problem: how to find analogues of the Baire category theorem for quasi-metric spaces. As a matter of fact, attempts in this direction have been made by several authors. For instance, Kelly [10] and Reilly et al [13] discovered a version of the Baire category theorem for quasi-metric spaces in a bitopological context; Ferrer and Gregori [7] showed that every quasi-regular left K-sequentially complete quasi-pseudo-metric space is a Baire space. Moreover, Ferrer and Gregori's result was re-discovered by Bentley et al in [3]. By studying Problem 1.2, we can improve all these existing analogues of the Baire category theorem.

2. Main Results

Recall that a topological space X is said to be *quasi-regular* if for every nonempty open set $V \subseteq X$, there is a nonempty open set U such that $\overline{U} \subseteq V$.

The following version of Baire category theorem for quasi-pseudo-metric spaces has been established in the literature.

Theorem 2.1 ([3], [7]). *A quasi-regular left K-complete quasi-pseudo-metric space (X, d) is a Baire space.*

Note that the "quasi-regular" cannot be dropped in Theorem 2.1, as shown in [7]. Using some Amsterdam property, we can improve Theorem 2.1 as shown in our next theorem.

Lemma 2.2. *A topological space X is weakly hypocompact if, and only if X admits a sequence of bases $\{\mathfrak{B}_n : n \in \mathbb{N}\}$ such that $\bigcap_{j \in \mathbb{N}} \overline{B_j} \neq \emptyset$ for every sequence $\{B_j : j \in \mathbb{N}\}$ satisfying $B_j \in \mathfrak{B}_{m_j}$ with $\{m_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$, $m_j < m_{j+1}$ and $B_{j+1} \subseteq B_j$ for all $j \in \mathbb{N}$.*

Proof. The necessity is trivially true. To show the sufficiency, let $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ be a countable filterbase such that $F_n \in \mathfrak{B}_{k_n}$ with $k_1 < k_2 < \cdots$. Put $U_1 = F_1$ and $n_1 := \min\{n > 1 : F_n \subseteq U_1\}$. Then, there is an $U_2 \in \mathcal{F}$ such that $U_2 \subseteq \bigcap_{i=1}^{n_1+1} F_i$. Put

$n_2 = \min\{n > n_1 : F_n \subseteq U_2\}$. Then $n_1 < n_2$. Repeating this process, we obtain a sequence $\{U_j : j \in \mathbb{N}\} \subseteq \mathcal{F}$ and a subsequence $\{F_{n_j} : j \in \mathbb{N}\} \subseteq \mathcal{F}$ which satisfy the following conditions

(i) $U_{j+1} \subseteq \bigcap_{i=1}^{n_j+1} F_i$ for all $j \in \mathbb{N}$,

(ii) $n_{j+1} = \min\{n > n_j : F_n \subseteq U_{j+1}\}$ for all $j \in \mathbb{N}$,

(iii) $F_{n_j} \in \mathfrak{B}_{k_{n_j}}$,

(iv) $k_{n_1} < k_{n_2} < \cdots$,

(v) $F_{n_j} \supseteq F_{n_{j+1}}$ for all $j \in \mathbb{N}$.

For convenience, we write $B_j = F_{n_j}$ and $m_j = k_{n_j}$ for all $j \in \mathbb{N}$. Then, by the assumption and our previous construction, we obtain $\bigcap_{n \in \mathbb{N}} \overline{F_n} = \bigcap_{j \in \mathbb{N}} \overline{B_j} \neq \emptyset$. This completes the proof of the sufficiency. $\qquad\square$

Theorem 2.3. *Every left K-sequentially complete quasi-pseudo-metric space is weakly hypocompact.*

Proof. Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space. For each $n \in \mathbb{N}$, let
$$\mathfrak{B}_n = \left\{ B_d\left(x, \frac{1}{2^k}\right) : k \geq n, x \in X \right\}.$$
Then, it is clear that $\{\mathfrak{B}_n : n \in \mathbb{N}\}$ is a sequence of bases for τ_d. To show that (X, d) is weakly hypocompact, by Lemma 2.2, let $\{B_j : j \in \mathbb{N}\}$ be a sequence such that $B_j \in \mathfrak{B}_{m_j}$ with $\{m_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$, $m_j < m_{j+1}$ and $B_{j+1} \subseteq B_j$ for all $j \in \mathbb{N}$. Without loss of generality, we may assume that $B_j = B_d\left(x_{m_j}, \frac{1}{2^{m_j}}\right)$ for each $j \in \mathbb{N}$. Since $\{m_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ is an increasing sequence, it is easily verified that $\{x_{m_j} : j \in \mathbb{N}\}$ is a left K-Cauchy sequence in (X, d). Since (X, d) is a left K-sequentially complete quasi-pseudo-metric space, $\{x_{m_j} : j \in \mathbb{N}\}$ converges to a point $x_0 \in X$. Clearly, $x_0 \in \bigcap_{j \in \mathbb{N}} \overline{B_j}$, and therefore (X, d) is weakly hypocompact. $\qquad\square$

Corollary 2.4. *If a quasi-pseudo-metric space has one of the following properties:*

(i) *left p-sequentially complete,*

(ii) *weakly left K-sequentially complete,*

(iii) *left K-sequentially complete,*

then it is a weakly hypocompact space, and thus a Baire space.

Proof. Note that left p-sequentially complete $\Rightarrow$ weakly left K-sequentially complete $\Rightarrow$ left K-sequentially complete, and then Theorem 2.3 applies. $\qquad\square$

Since every quasi-regular hypocompact space is Baire, Theorem 2.1 follows directly from Theorem 2.3. To strengthen the conclusion of Theorem 2.3, we need an auxiliary concept and lemma. Let X be a nonempty set. We say that a family $\mathcal{F}$ of subsets of X has the *decreasing chain condition* [9] if any properly decreasing sequence of elements of $\mathcal{F}$ is finite.

Lemma 2.5 ([9]). *Any cover of a nonempty set X has a subcover which has the decreasing chain condition.*

Recall that a space X is said to be *quasi-developable* if there is a sequence of families of nonempty open subsets $\{\mathfrak{G}_n : n \in \mathbb{N}\}$ such that for each $x \in X$, $\{\mathrm{st}(x, \mathfrak{G}_n) : \mathrm{st}(x, \mathfrak{G}_n) \neq \emptyset$, and $n \in \mathbb{N}\}$ is a local base at x, where

$$\mathrm{st}(x, \mathfrak{G}_n) = \bigcup \{G \in \mathfrak{G}_n : x \in G\}.$$

for $x \in X$ and $n \in \mathbb{N}$.

Theorem 2.6. *Every regular, quasi-developable and left K-sequentially complete quasi-pseudo-metric space is hypocompact.*

Proof. Let (X, d) be a regular, quasi-developable and left K-sequentially complete quasi-pseudo-metric space. For each $n \in \mathbb{N}$, let

$$\mathfrak{B}_n = \left\{ B_d \left(x, \frac{1}{2^k} \right) : k \geq n, x \in X \right\}.$$

Then $\{\mathfrak{B}_n : n \in \mathbb{N}\}$ is a sequence of bases for τ_d on X. First, we verify that if $\{B_n : n \in \mathbb{N}\}$ is a properly decreasing sequence of open subsets of X with $B_n \in \mathfrak{B}_n$ for all $n \in \mathbb{N}$, then $\{\overline{B_n} : n \in \mathbb{N}\}$ is a convergent filterbase. Again, for each $n \in \mathbb{N}$, we assume that $B_n = B_d(x_{k_n}, \frac{1}{2^{k_n}})$ for some $k_n \geq n$, where $k_1 < k_2 < \cdots$. Then, as a left K-Cauchy sequence in (X, d), $\{x_{k_n} : n \in \mathbb{N}\}$ converges to a point $x_0 \in X$. We claim that $\{\overline{B_n} : n \in \mathbb{N}\}$ converges to x_0. Suppose the contrary. Then there exist an open neighbourhood V of x_0 and a sequence $\{y_{k_n} : n \in \mathbb{N}\}$ in X such that $y_{k_n} \in B_d(x_{k_n}, \frac{1}{2^{k_n}}) \setminus V$ for all $n \in \mathbb{N}$. Observe that $\{y_{k_n} : n \in \mathbb{N}\}$ also converges to the point x_0. As $\{y_{k_n} : n \in \mathbb{N}\} \subseteq X \setminus V$, $x_0 \notin V$. This is a contradiction.

Now, let $\{\mathfrak{G}_n : n \in \mathbb{N}\}$ be a quasi-development for X. Define

$$\mathfrak{B}'_n = \{B \in \mathfrak{B}_n : \overline{B} \subseteq G \text{ for some } G \in \mathfrak{G}_n\}.$$

For every $n \in \mathbb{N}$, by Lemma 2.5, there is a subfamily $\mathfrak{B}''_n$ of $\mathfrak{B}'_n$ such that

(i) $\bigcup \mathfrak{B}'_n = \bigcup \mathfrak{B}''_n$;

(ii) $\mathfrak{B}''_n$ satisfies the decreasing chain condition.

We first claim that $\bigcup_{n \in \mathbb{N}} \mathfrak{B}''_n$ is a base for X. In fact, for each $x \in X$ and each neighborhood U of x, there is some $n_0 \in \mathbb{N}$ such that $\{G \in \mathfrak{G}_{n_0} : x \in G\} \neq \emptyset$ and $\operatorname{st}(x, \mathfrak{G}_{n_0}) \subseteq U$. Thus, $x \in \bigcup \mathfrak{B}'_{n_0} = \bigcup \mathfrak{B}''_{n_0}$. Choose $B \in \mathfrak{B}''_{n_0}$ with $x \in B$, then $B \subseteq \operatorname{st}(x, \mathfrak{G}_{n_0}) \subseteq U$. To show that X is hypocompact, let $\mathcal{F} \subseteq \bigcup_{n \in \mathbb{N}} \mathfrak{B}''_n$ be a filterbase. Suppose that $\mathcal{F}$ does not have a cluster point, i.e., $\bigcap_{F \in \mathcal{F}} \overline{F} = \emptyset$. First, pick an arbitrary $F_1 \in \mathcal{F}$. Then, there must be some element $F'_2 \in \mathcal{F}$ with $F_1 \not\subseteq F'_2$. Choose a element $F_2 \in \mathcal{F}$ such that $F_2 \subseteq F_1 \cap F'_2$. It is clear that F_2 is a proper subset of F_1. Continuing this process, we obtain a properly decreasing sequence $F_1 \supset F_2 \supset \cdots$ in $\mathcal{F}$ (and thus in $\bigcup_{n \in \mathbb{N}} \mathfrak{B}''_n$). By (ii), we can take a properly increasing subsequence $k_1 < k_2 < \cdots$ such that $F_n \in \mathfrak{B}''_{k_n}$ for all $n \in \mathbb{N}$. As pointed out previously, $\{\overline{F_n} : n \in \mathbb{N}\}$ must converge to some point $p \in X$. Since $p \notin \bigcap_{F \in \mathcal{F}} \overline{F}$, then $p \in X \setminus \overline{F_p}$ for some $F_p \in \mathcal{F}$. Hence, there is some $k \in \mathbb{N}$ such that $F_k \subseteq X \setminus \overline{F_p}$. It follows that $F_k \cap F_p = \emptyset$. This contradicts with the fact that $\mathcal{F}$ is a filterbase. Therefore, we have $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$ and X is hypocompact. □

Corollary 2.7. *If a regular and quasi-developable quasi-pseudo-metric space has any of the following properties*

(i) *left p-sequentially complete,*

(ii) *weakly left K-sequentially complete,*

then it is hypocompact.

Corollary 2.8 ([14]). *Every regular, quasi-developable and left K-sequentially complete quasi-metric space (X, d) is subcompact.*

The answers of the following two questions are still unclear to the authors.

Question 2.9. *Can the conclusion in Theorem 2.6 be sharpened to be base-compact?*

Question 2.10. *Must every left K-sequentially complete quasi-pseudo-metric space be base-compact, subcompact or hypocompact?*

3. Some Examples

In this section, we consider the converse of Problem 1.2, that is, if a quasi-metric space has some Amsterdam properties, must it be completely quasi-metrizable in some sense?

It is shown in [9] that a metrizable space is completely metrizable if, and only if it is subcompact. However, until now, it seems to the authors that no quasi-metric analogue to this result has been found yet.

Lemma 3.1 ([14]). *A regular paracompact space is completely metrizable if, and only if it has a compatible left K-sequentially complete quasi-metric.*

Example 3.2. *The Sorgenfrey line $\mathbb{S}$ is quasi-metrizable and subcompact, but admits no compatible left K-sequentially complete quasi-metric.* It is known that $\mathbb{S}$ is a Hausdorff, subconmpact and paracompact space, which is neither base-compact nor metrizable. Hence, it admits no compatible left K-sequentially complete quasi-metric. Otherwise, by Lemma 3.1, it will be completely metrizable, which is a contradiction.

Another well-known fact is that a (pseudo-) metrizable space is base-compact if, and only if it is completely (pseudo-) metrizable, refer to [5, p. 401].

Example 3.3. *The Michael line $\mathbb{L}$ is quasi-metrizable and base-compact, but admits no compatible left K-sequentially complete quasi-metric.* It is known that $\mathbb{L}$ is a Hausodrff, base-compact, paracompact space which is not metrizable. Thus, by Lemma 3.1, $\mathbb{L}$ admits no compatible left K-sequentially complete quasi-metric.

In what follows, we provide another example which has the properties similar to those of spaces mentioned in Example 3.2 and Example 3.3. To this purpose, we need to introduce some notation. Let V be a relation on a topological space X, that is, $V \subseteq X \times X$. Recall that V is said to be *transitive* if $V \circ V \subseteq V$. Furthermore, V is called a *neighbornet* if for every $x \in X$, $V(x) := \{y \in X : (x, y) \in V\}$ is a neighborhood of x.

Lemma 3.4 ([8]). *A T_1 topological space X is non-Archimedeanly quasi-metrizable if and only if there is a sequence $\{T_n : n \in \mathbb{N}\}$ of transitive neighbournets such that $\{T_n(x) : n \in \mathbb{N}\}$ is a local base for every point $x \in X$.*

Example 3.5. *A base-compact quasi-metrizable space which admits no compatible left K-sequentially complete quasi-metric.* Let $X = \bigcup_{\alpha \leqslant \omega_1} \mathbb{N}^\alpha$. For each $\alpha < \omega_1$, $x \in \mathbb{N}^\alpha$ and $n \in \mathbb{N}$, put

$$U_n(x) = \{x\} \cup \{y \in X : x \subseteq y \text{ and } y(\alpha) \geq n\};$$

and for each $x \in \mathbb{N}^{\omega_1}$, let $U_n(x) = \{x\}$ for all $n \in \mathbb{N}$. Let X be equipped with the topology generated by the base

$$\mathfrak{B} = \{U_n(x) : x \in X, \text{ and } n \in \mathbb{N}\}.$$

First, we show that X is non-Archimedeanly quasi-metizable and non-metrizable. For each $n \in \mathbb{N}$, set $T_n = \bigcup_{x \in X} \{x\} \times U_n(x)$. Since $T_n \circ T_n \subseteq T_n$ for each $n \in \mathbb{N}$, then $\{T_n : n \in \mathbb{N}\}$ is a sequence of transitive neighbournets. By Lemma 3.4, X is non-Archimedeanly quasi-metrizable. Since the subspace $\bigcup_{\alpha < \omega_1} \mathbb{N}^\alpha$ of X is precisely the space defined in [2], which is non-metrizable, we conclude that X is non-metrizable either.

Secondly, we show that X is base-compact. Observe that for any two elements $B_1, B_2 \in \mathfrak{B}$, either $B_1 \cap B_2 = \emptyset$, or one of them contains the other. In fact, suppose that $B_1 = U_n(x)$ and $B_2 = U_m(y)$. If neither of x and y is an extension of the other, then $U_n(x) \cap U_m(y) = \emptyset$. If x is an extension of y, but $x \notin U_m(y)$, we still have $U_n(x) \cap U_m(y) = \emptyset$. Finally, if $x \in U_m(y)$ then $U_n(x) \subseteq U_m(y)$. Now let $\mathcal{F} \subseteq \mathfrak{B}$ be a subfamily which has the finite intersection property. Then $\mathcal{F}$ is linearly ordered by set-theoretic inclusion $\subseteq$. Enumerate $\mathcal{F}$ as $\mathcal{F} = \{U_{n(\alpha)}(x_\alpha) : \alpha \in A\}$. It is easy to see that $x = \bigcup_{\alpha \in A} x_\alpha \in \bigcap_{F \in \mathcal{F}} \overline{F}$. It follows that X is base-compact.

Next, we prove that X is paracompact. Let $\mathcal{U}$ be an open cover of X. For each point $x \in \mathbb{N}^2$, there is a $U_{n(x)}(x) \in \mathfrak{B}$ contained in some element of $\mathcal{U}$. Let $\mathcal{V}_2 = \{U_{n(x)}(x) : x \in \mathbb{N}^2\}$. Since no two points of $\mathbb{N}^2$ are extensions of each other, $\mathcal{V}_2$ is pairwise disjoint. For each $\beta < \omega_1$, suppose that we have constructed a disjoint subfamily $\mathcal{V}_\alpha \subseteq \mathfrak{B}$ such that $\mathcal{V}_\alpha$ is a cover of $\bigcup_{\gamma \leqslant \alpha} \mathbb{N}^\gamma$, $\mathcal{V}_\gamma \subseteq \mathcal{V}_\alpha$ if $\gamma < \alpha < \beta$, and each element of $\mathcal{V}_\alpha$ is contained in some element of $\mathcal{U}$. Now choose some $U_{n(x)}(x) \in \mathfrak{B}$ which is contained in some element of $\mathcal{U}$ for each point $x \in \mathbb{N}^\beta \setminus \bigcup_{\alpha < \beta} \bigcup \mathcal{V}_\alpha$. Clearly,

$$U_{n(x)}(x) \cap \left(\bigcup_{\alpha < \beta} \bigcup \mathcal{V}_\alpha \right) = \emptyset.$$

Now, put

$$\mathcal{V}_\beta = \{U_{n(x)}(x) : x \in \mathbb{N}^\beta\} \cup \bigcup_{\alpha < \beta} \mathcal{V}_\alpha,$$

and

$$\mathcal{V} = \bigcup_{\alpha < \omega_1} \mathcal{V}_\alpha \cup \left\{ \{x\} : x \in X \setminus \bigcup_{\alpha < \omega_1} \mathcal{V}_\alpha \right\}.$$

Then, $\mathcal{V}$ is a pairwise disjoint open refinement of $\mathcal{U}$. Hence, X is paracompact.

Finally, if X admits a left K-sequentially complete quasi-metric, by Lemma 3.1, it is completely metrizable. This is a contradiction. Therefore, X admits no compatible left K-sequentially complete quasi-metric. $\square$

Question 3.6. *Must every Čech-complete quasi-metric space admit a compatible left K-sequentially complete quasi-metric?*

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