Wave Approach in Discrete-Continuous Systems Longitudinally Deformed with Variable Rod Cross-Sections

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Abstract. The paper deals with the dynamics of discrete-continuous systems longitudinally deformed. These systems consist of elastic elements connected by means of rigid bodies. In the discussion a wave method using the solution of the d'Alembert type is applied, what leads to equations with a retarded argument. After a general information concerning the wave approach, detailed considerations are done for a system consisting of three rods and two rigid bodies. Rods in the system have variable cross-sections and in numerical calculations their effect on rods' displacements is investigated.

Keywords: Dynamical systems, rods, discrete-continuous models, waves, variable cross-section.

AMS Subject Classification: 34C20, 34C25, 93C10.

1. INTRODUCTION

In the paper dynamical discrete-continuous systems are investigated. Such systems consist of elastic elements connected by means of rigid bodies. The considered systems belong to a certain class of discrete-continuous systems, namely to those where the motion of elastic elements with a constant cross-section is described by the classical wave equation. In the discussion a wave approach is used, what leads to solving equations with a retarded argument. The proposed approach can be applied in the study of systems longitudinally and torsionally deformed, strings or systems subject to shear deformations, [9]. Elastic elements in the systems can have variable or constant cross-sections. It is required that functions describing variable cross-sections allow to use the solution of the d'Alembert type for equations of motion, i.e., to use the wave approach.

At the beginning, the wave approach is presented shortly. Detailed considerations are done next for a system consisting of two rigid bodies and three noncoaxial rods longitudinally deformed with variable cross-sections. Such systems can be used in the modelling of fragments of plane trusses, [3,4,7,8]. The rod-rigid element systems with constant rod cross-sections are discussed in [7,8,11]. In [7] linear systems while in [8,11] systems with a local nonlinearity are studied.

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The aim of the present paper is to propose functions for the description of rod cross-sections, to formulate the initial-boundary problem for the linear system consisting of two rigid bodies and three rods having variable cross-sections, and to investigate in numerical calculations the effect of the variable cross-sections on the behavior of the considered system.

2. WAVE APPROACH

The proposed wave approach can be applied in the discussion of multi-mass discrete-continuous systems subject to longitudinal, torsional or shear deformations. These systems consist of elastic elements connected by rigid bodies. During the motion the cross-sections of elastic elements remain flat. Rigid bodies can be loaded by an external loading $P(t)$. Elastic elements can have variable cross-sections described by the functions

$$A_i(x) = A_{0i}\left(1 - x / b_{0i}\right)^2 \tag{2.1}$$

where $A_i(x) = 0$ for $x = b_{0i}$ and $A_i(x) = A_{0i}$ is constant for $b_{0i} \to \pm\infty$. Exemplary diagrams of functions (2.1) are given in Fig. 1 for i = 1. From Fig. 1 it follows that for $b_{0i} = -1000$ functions (2.1) can be considered as constant functions. It should be pointed out that functions (2.1) allow to apply the solutions of the d'Alembert type for equations of motion, i.e., to use the wave approach.

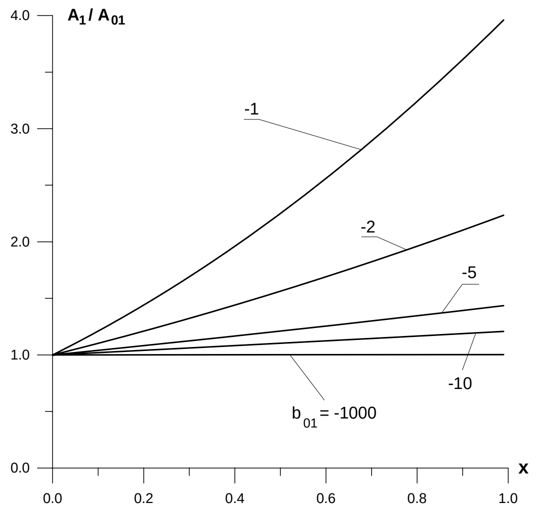

Fig. 1. Diagrams of functions (2.1)

The determination of displacements W_i of the *i*th elastic element having the cross-sections (2.1) is reduced to solving the following equation of motion

$$\frac{\partial^2 W_i}{\partial t^2} - a^2\left(\frac{\partial^2 W_i}{\partial x^2} - \frac{2}{b_{0i} - x}\frac{\partial W_i}{\partial x}\right) = 0, \qquad i = 1,2,3, \tag{2.2}$$

with a being a wave speed. The solutions of equations (2.2) are sought in the form, [5],

$$W_i(x,t) = \frac{1}{x - b_{0i}}\left[f_i(a(t - t_{0i}) - x + x_{0i}) + g_i(a(t - t_{0i}) + x - x_{0i})\right] \tag{2.3}$$

where f_i and g_i represent waves propagating in the *i*th elastic element in the consistent and opposite to x-axis senses, and the constants t_{0i} and x_{0i} denote the time instant and the end of the *i*th elastic element where the first disturbance caused by the external force $P(t)$ occurs. For the constant cross-section, equations (2.2) become classical wave equations and solutions are looking for only by means of the sum of the functions f_i and g_i, i.e., without denominators in (2.3), and with the same arguments. The assumed solutions satisfy identically the equations of motion (2.2).

To equations (2.2) one has to add zero initial conditions and appropriate boundary conditions which are conditions for displacements and forces acting in the cross-sections where rigid bodies are located. Upon substituting (2.3) into appropriate boundary conditions the equations with a retarded argument are obtained, [7-11]. Differential-difference equations have incessantly some attention in the literature, [1,2,6]. It appears that equations of such a type are obtained in the study of discrete-continuous dynamical systems. Details investigations are presented below for the system longitudinally deformed.

3. SYSTEM LONGITUDINALLY DEFORMED, GOVERNING EQUATIONS

The wave approach can be applied for various systems having elastic elements described by equations (2.2). Here this approach is shown for the linear system consisting of 3 rods longitudinally deformed and of 2 rigid bodies, Fig. 2. Rods are not coaxial, so in the description a fixed reference system $0xy$ and one-dimensional coordinate systems $0_i x_i$ assigned to individual *i*th rod are used. The displacement of the cross-section x_i in the *i*th rod is described by the function $u_i(x_i, t)$ dependent on the location of the considered cross-section and on time, whereas the time functions U_j, V_j are the components of the displacements of the *j*th rigid body in the *x*-axis and *y*-axis directions, respectively.

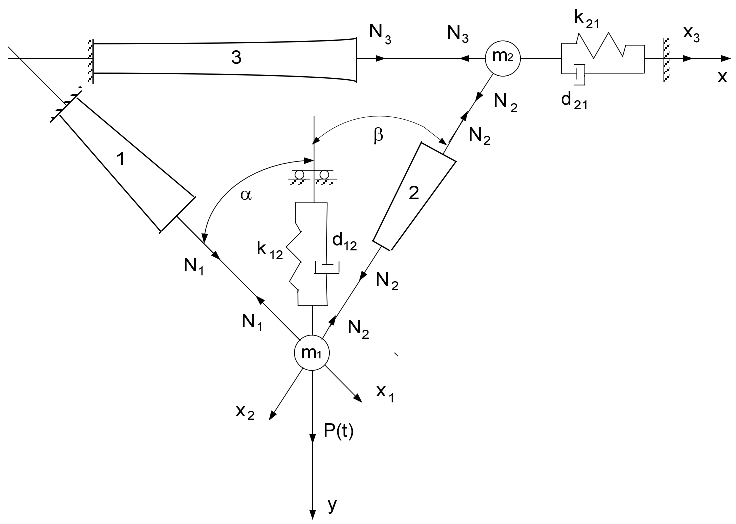

Fig. 2. Rod-rigid element system longitudinally deformed

The considered system can represent a fragment of plane trusses with members subject to longitudinal deformations modelled by discrete-continuous models, [3,4,7,8]. An external excitation is applied to a rigid body located, for example, in a truss joint. In this cross-section a visco-elastic discrete element is also located. Rods are characterized by the Young's modulus *E*, density ρ and the length l_i. The cross-section of rods can be variable. We proposed to describe them by functions (2.1). Damping in the rods is taken into account by means of an equivalent internal and external damping with appropriate coefficients D_i and d_{ij}. Here the case when the motion of the rigid body having mass m_1 in the *x*-axis direction is neglected ($U_1 = 0$) is discussed.

It is convenient to perform the discussion in the following nondimensional quantities

$$\bar{u}_i = u_i / u_0, \quad \bar{x}_i = x_i / l_0, \quad \bar{t} = at / l_0, \quad \bar{l}_i = l_i / l_0, \quad \bar{D}_i = D_i a / l_0,$$

$$\overline{d}_{ij} = d_{ij} l_0 /(a m_0)\,, \quad R_i = m_i / m_0\,, \quad K_i = A_{0i}\rho l_0 / m_0\,, \quad \overline{A}_i = A_i / A_{0i}\,, \tag{3.1}$$

$$\overline{b}_{0i} = b_{0i} / l_0\,, \quad \overline{P} = P l_0^2 /(u_0 m_0 a^2)\,, \quad \overline{k}_{ij} = k_{ij} l_0^2 /(m_0 a^2)$$

where m_0, l_0 and u_0 are fixed mass, length and displacement, respectively.

Under the assumed assumptions, the determination of nondimensional displacements $u_i(x_i, t)$ of rods leads to solving three equations of motion

$$\frac{\partial^2 u_i}{\partial t^2} - a^2 \left(\frac{\partial^2 u_i}{\partial x_i^2} - \frac{2}{b_{0i} - x_i} \frac{\partial u_i}{\partial x_i} \right) = 0 \qquad i = 1,2,3, \tag{3.2}$$

where $a^2 = E / \rho$, with zero initial conditions

$$u_i(x_i, t) = \frac{\partial u_i}{\partial t}(x_i, t) = 0 \quad \text{for } t = 0, \quad i = 1,2,3, \tag{3.3}$$

and with the following boundary conditions

$$u_1(x_1, t) = 0 \quad \text{for} \quad x_1 = 0\,,$$

$$u_1 \cos\beta - u_2 \cos\alpha = 0 \quad \text{for} \quad x_1 = l_1, \quad x_2 = l_2\,,$$

$$\begin{aligned} & - R_1 \left(C_2 \frac{\partial^2 u_1}{\partial t^2} + C_1 \frac{\partial^2 u_2}{\partial t^2} \right) - d_{12} \left(C_2 \frac{\partial u_1}{\partial t} + C_1 \frac{\partial u_2}{\partial t} \right) + \\ & \qquad - k_{12}\left(C_2 u_1 + C_1 u_2\right) - A_1(x_1) K_1 \cos\alpha \left(D_1 \frac{\partial^2 u_1}{\partial x_1 \partial t} + \frac{\partial u_1}{\partial x_1} \right) + \\ & \qquad - A_2(x_2) K_2 \cos\beta \left(D_2 \frac{\partial^2 u_2}{\partial x_2 \partial t} + \frac{\partial u_2}{\partial x_2} \right) + P(t) = 0 \quad \text{for} \quad x_1 = l_1, \quad x_2 = l_2\,, \\ & - R_2 \frac{\partial^2 u_3}{\partial t^2} - d_{21} \frac{\partial u_3}{\partial t} - k_{21} u_3 - A_3(x_3) K_3 \left(D_3 \frac{\partial^2 u_3}{\partial x_3 \partial t} + \frac{\partial u_3}{\partial x_3} \right) + \\ & \qquad - A_2(x_2) K_2 \sin\beta \left(D_2 \frac{\partial^2 u_2}{\partial x_2 \partial t} + \frac{\partial u_2}{\partial x_2} \right) = 0 \quad \text{for} \quad x_2 = 0, \quad x_3 = l_3 \end{aligned} \tag{3.4}$$

$$u_3 \sin\beta + u_2 = 0 \quad \text{for} \quad x_2 = 0, \;\; x_3 = l_3,$$

$$u_3(x_3, t) = 0 \quad \text{for} \quad x_3 = 0$$

with $C_1 = \sin\alpha / \sin(\alpha+\beta)$, $C_2 = \sin\beta / \sin(\alpha+\beta)$. In relations (3.2) - (3.4) bars denoting nondimensional quantities are omitted, for convenience. When the rods in the considered system have constant cross-sections then the boundary conditions (3.4) coincide with appropriate boundary conditions given in [7].

The solutions for equations (3.2) are looking for in the form

$$u_1(x_1, t) = \frac{1}{x_1 - b_{01}}\left[f_1(t - x_1 + l_1) + g_1(t + x_1 - l_1) \right],$$

$$u_2(x_2, t) = \frac{1}{x_2 - b_{02}}\left[f_2(t - x_2 + l_2) + g_2(t + x_2 - l_2) \right], \tag{3.5}$$

$$u_3(x_3, t) = \frac{1}{x_3 - b_{03}}\left[f_3(t - x_3 - l_2 + l_3) + g_3(t + x_3 - l_2 - l_3) \right].$$

In seeking solutions it is taken into account that the first disturbance caused by the external force $P(t)$ occurs in rods (1) and (2) at time instant $t = 0$ in $x_1 = l_1$, $x_2 = l_2$, while in the rod (3) at $t = l_2$ in $x_3 = l_3$.

Upon substituting (3.5) into boundary conditions (3.4) the following equations with a retarded argument for unknown functions $f_i(z)$ and $g_i(z)$ are obtained

$$f_1(z) = -g_1(z - 2l_1),$$

$$f_2(z) = -g_2(z - 2l_2) + b_{02} L_3 \sin\beta\left[f_3(z - 2l_2) + g_3(z - 2l_2) \right],$$

$$f_3(z) = -g_3(z - 2l_3), \tag{3.6}$$

$$r_{11} g_1''(z) = r_{12} g_1'(z) + r_{13} f_1''(z) + r_{14} f_1'(z) + r_{15} f_2''(z) + r_{16} f_2'(z) + r_{17}\left[f_1(z) + g_1(z) \right] + L_1^{-1} P(z),$$

$$g_2(z) = -f_2(z) + L_1 L_2^{-1}\left[f_1(z) + g_1(z) \right]\cos\beta / \cos\alpha,$$

$$r_{31} g_3''(z) = r_{32} g_3'(z) + r_{33} f_3''(z) + r_{34} f_3'(z) + r_{35} g_2''(z) + r_{36} g_2'(z) + r_{37}\left[f_3(z) + g_3(z) \right]$$

where

$$
\begin{aligned}
r_{11} &= R_1 / \cos\alpha + S_1 K_1 D_1 \cos\alpha + S_2 K_2 D_2 \cos^2\beta / \cos\alpha\,, \\
r_{12} &= S_1 K_1 D_1 L_1 \cos\alpha - d_{12} / \cos\alpha - S_1 K_1 \cos\alpha + S_2 K_2 D_2 L_2 \cos^2\beta / \cos\alpha \\
&\qquad - S_2 K_2 \cos^2\beta / \cos\alpha\,, \\
r_{13} &= S_1 K_1 D_1 \cos\alpha - R_1 / \cos\alpha - S_2 K_2 D_2 \cos^2\beta / \cos\alpha\,, \\
r_{14} &= S_1 K_1 D_1 L_1 \cos\alpha - d_{12} / \cos\alpha + S_1 K_1 \cos\alpha + S_2 K_2 D_2 L_2 \cos^2\beta / \cos\alpha \\
&\qquad - S_2 K_2 \cos^2\beta / \cos\alpha\,, \\
r_{15} &= 2 S_2 K_2 D_2 L_1^{-1} L_2 \cos\beta\,, \quad r_{16} = 2 S_2 K_2 L_1^{-1} L_2 \cos\beta\,, \\
r_{17} &= S_1 K_1 L_1 \cos\alpha - k_{12} / \cos\alpha + S_2 K_2 L_2 \cos^2\beta / \cos\alpha\,, \\
r_{31} &= R_2 + S_3 K_3 D_3 + K_2 D_2 \sin^2\beta\,, \\
r_{32} &= S_3 K_3 D_3 L_3 - d_{21} - S_3 K_3 + K_2 D_2 \sin^2\beta / b_{02} - K_2 \sin^2\beta\,, \\
r_{33} &= S_3 K_3 D_3 - R_2 - K_2 D_2 \sin^2\beta\,, \\
r_{34} &= S_3 K_3 D_3 L_3 - d_{21} + S_3 K_3 + K_2 D_2 \sin^2\beta / b_{02} - K_2 \sin^2\beta\,, \\
r_{35} &= 2 K_2 D_2 L_3^{-1} \sin\beta / b_{02}\,, \quad r_{36} = 2 K_2 L_3^{-1} \sin\beta / b_{02}\,, \\
r_{37} &= S_3 K_3 L_3 - k_{21} + K_2 \sin^2\beta / b_{02}\,, \\
L_i &= (l_i - b_{0i})^{-1}\,, \quad S_i = (1 - l_i / b_{0i})^2\,, \quad i = 1,2,3.
\end{aligned}
\tag{3.7}
$$

Equations (3.6) are linear. Differential equations in (3.6) are solved numerically by means of the Runge-Kutta method.

The problem similar to (3.2) - (3.4), but not in details, is formulated in [10] for the system consisting of two rods having variable cross-sections and of two rigid bodies with a local nonlinearity. The equations with a retarded argument obtained there are nonlinear. Though these equations takes into account variable cross-sections, presented exemplary numerical results are done for $b_{0i} = -1000$, i.e. for constant rod cross-sections (see Fig. 1).

4. Numerical results

In numerical calculations the following dimensional parameters are assumed

$$
\begin{gathered}
l_0 = l_1 = l_2 = l_3 = 2\text{m}, \quad A_{0i} = 2\cdot 10^{-3}\,\text{m}^2, \quad \rho = 0.8\cdot 10^4\,\text{kg}/\text{m}^3, \\
E = 2.1\cdot 10^{11}\,\text{N/m}^2, \quad k_{ij} = 2.1\cdot 10^8\,\text{N/m}, \quad m_1 = 20\text{kg}, \\
m_2 = 3.2\text{kg}, \quad m_0 = 32\text{kg}, \quad a = 5000\text{m/s}, \\
P_0 = 200\text{kN}, \quad u_0 = 10^{-3}\,\text{m}, \quad \alpha = \beta = \pi/6\,.
\end{gathered}
\tag{4.1}
$$

Using (3.1) nondimensional parameters then are

$$
R_1 = 0.625, \quad R_2 = 0.1, \quad l_i = 1.0, \quad K_i = 1.0, \quad P_0 = 1.0, \quad k_{ij} = 1.05. \tag{4.2}
$$

Remaining parameters can change.

External loading occurring in (3.6) can be described by an arbitrary time function. Here it is represented by the function changing harmonically in time

$$
P(t) = P_0 \sin(pt) \tag{4.3}
$$

where p is the nondimensional frequency of the external force, and P_0 is its non-dimensional amplitude. Such a type of the external loading is important in engineering applications. Numerical solutions are focused on the determination of displacements in selected cross-sections in transient as well as in steady states of motion. All calculations are done with the nondimensional damping coefficients $d_{ij} = D_i = 0.2$.

The effect of variable cross-sections is presented in Figs 3 - 14 showing diagrams of appropriate displacements in transient, and next, in steady states. Figs 3 - 5 and Figs 9 - 11 concern the displacements $V_1 = C_2 u_1 + C_1 u_2$ of the rigid body m_1 in y-axis direction located in $x_1 = l_1$, $x_2 = l_2$ (see Fig. 2) with the frequency p equal 0.5 or 1.3, respectively. The first natural frequency of the considered system with parameters (4.2) is equal ω_1 = 1.369, [8], so p = 0.5 corresponds to results far while p = 1.3 corresponds to results close to the first resonant region. Figs 6 - 8 and Figs 12 - 14 concern the displacements u_3 of the rigid body m_2 in x-axis direction with $x_3 = l_3$, also for p = 0.5 and p = 1.3. In calculations three values of the parameters b_{0i} representing variable cross-sections (2.1) are assumed, i.e., b_{0i} = -1000, -20, -10. From

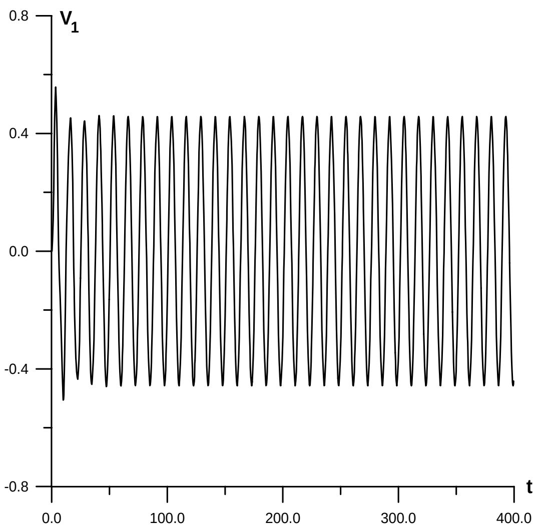

Fig. 3. Displacement V_1 of the rigid body m_1 for p = 0.5, b_{0i} = -1000

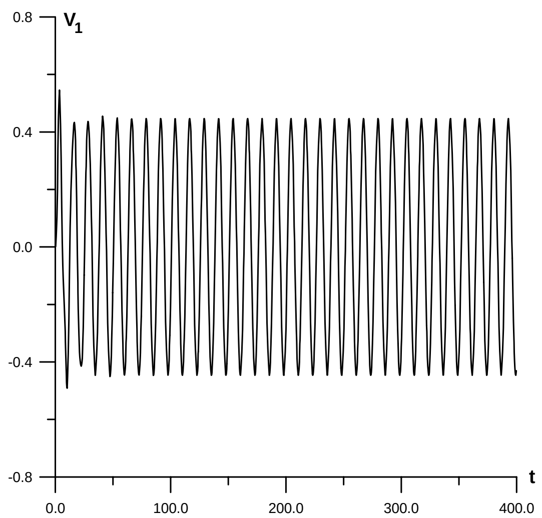

Fig. 4. Displacement V_1 of the rigid body m_1 for p = 0.5, b_{0i} = -20

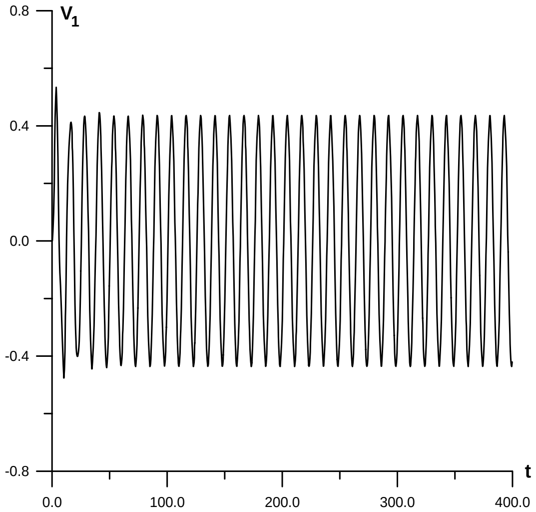

Fig. 5. Displacement V_1 of the rigid body m_1 for p = 0.5, b_{0i} = -10

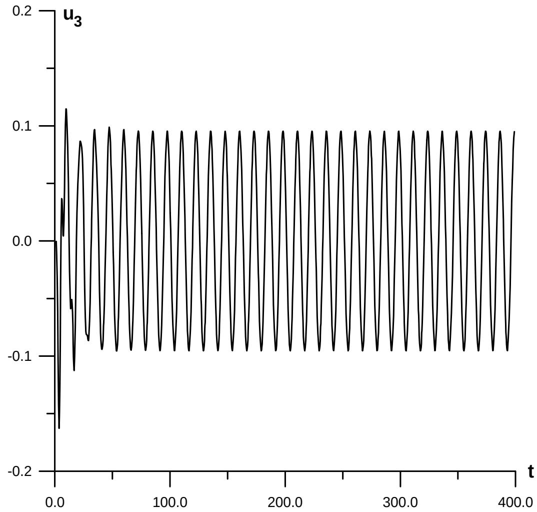

Fig. 6. Displacement u_3 for $x_3 = l_3$, p = 0.5, b_{0i} = -1000

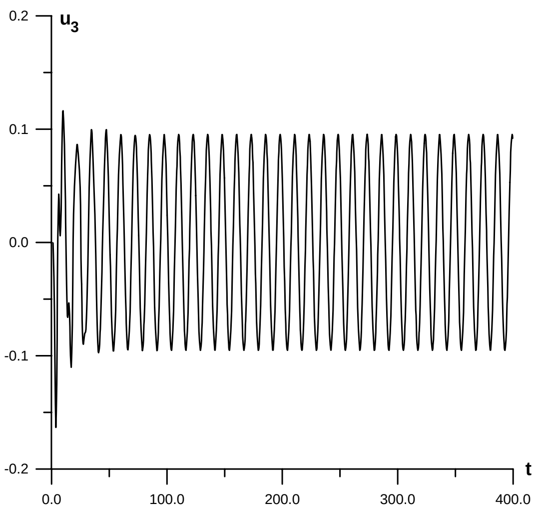

Fig. 7. Displacement u_3 for $x_3 = l_3$, p = 0.5, b_{0i} = -20

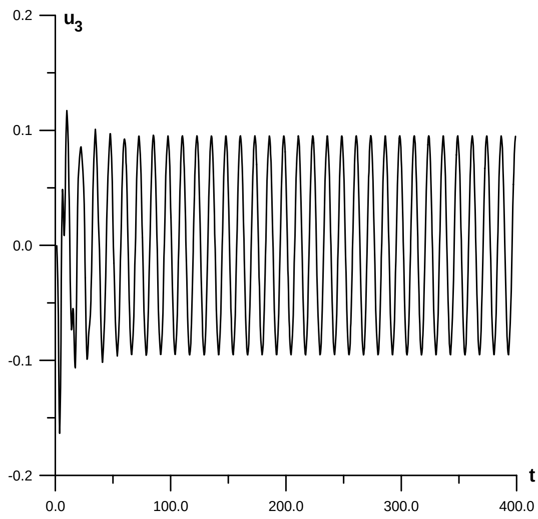

Fig. 8. Displacement u_3 for $x_3 = l_3$, p = 0.5, b_{0i} = -10

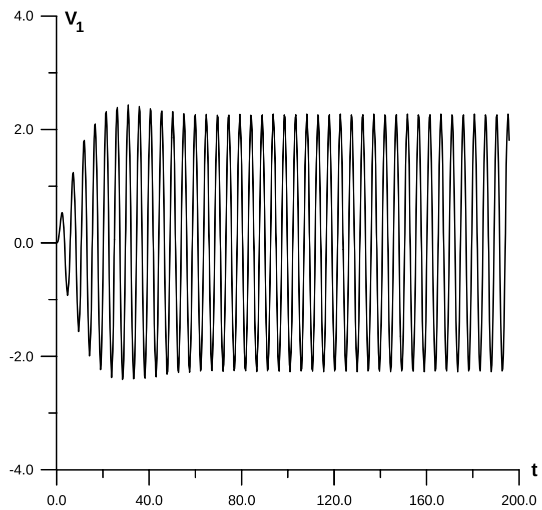

Fig. 9. Displacement V_1 of the rigid body m_1 for $p = 1.3$, $b_{0i} = -1000$

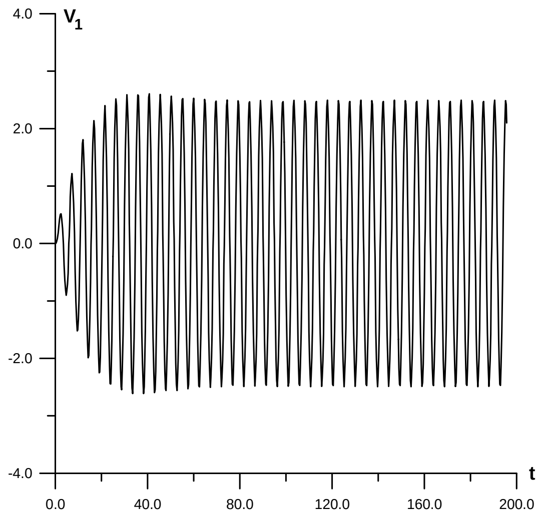

Fig. 10. Displacement V_1 of the rigid body m_1 for $p = 1.3$, $b_{0i} = -20$

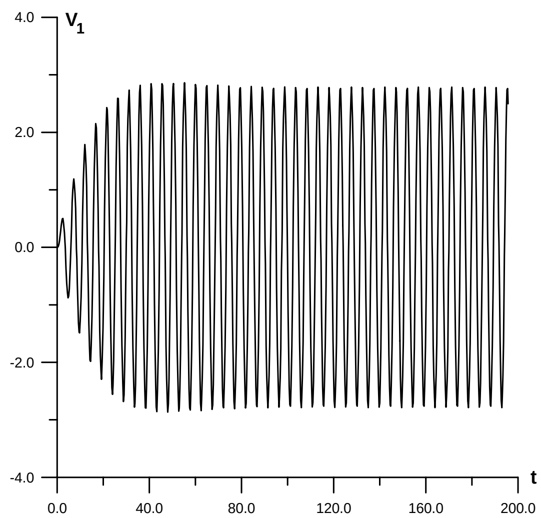

Fig. 11. Displacement V_1 of the rigid body m_1 for $p = 1.3$, $b_{0i} = -10$

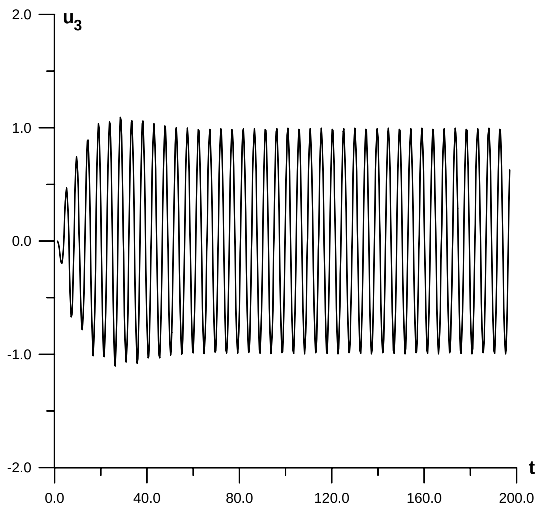

Fig. 12. Displacement u_3 for $x_3 = l_3$, $p = 1.3$, $b_{0i} = -1000$

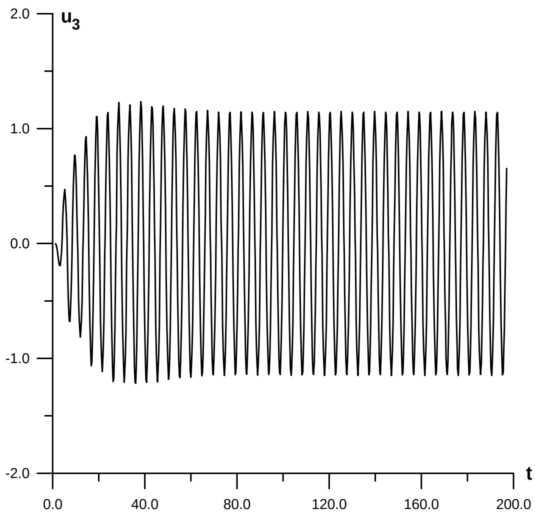

Fig. 13. Displacement u_3 for $x_3 = l_3$, p = 1.3, b_{0i} = -20

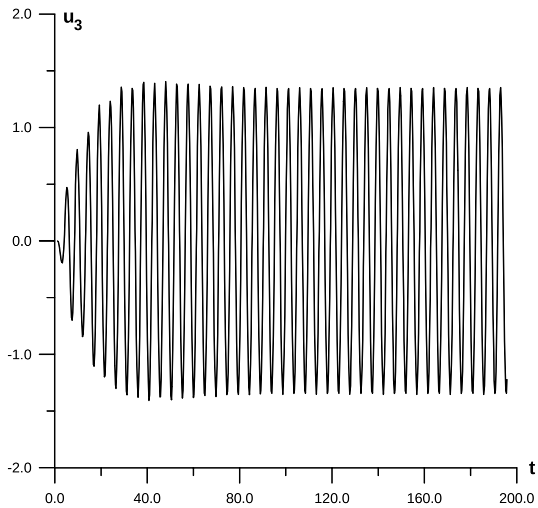

Fig. 14. Displacement u_3 for $x_3 = l_3$, p = 1.3, b_{0i} = -10

Figs 3 - 14 it follows that the solutions gain steady states earlier for p = 0.5. The strong effect of variable cross-sections is observed for p = 1.3, i.e. near the first resonant region. This concerns displacements V_1 as well as displacements u_3 and they increase then with the increase of b_{0i}.

Equations (3.6) using (3.5) allow to determine required solutions in arbitrary cross-sections of rods in the analyzed system. For this reason, in Fig. 15 displacements $u_i(x_i, t)$ in the cross-sections $x_1 = x_2 = x_3$ = 0.5 of rods (1), (2), (3) in the regions of the steady states of motion are presented for *a*) p = 0.5 and *b*) p = 1.3. From these diagrams it follows that the smallest displacement amplitudes are observed in the rod (3), i.e., for x_3 = 0.5, in the both cases of the frequency p of the external loading.

a) *b*)

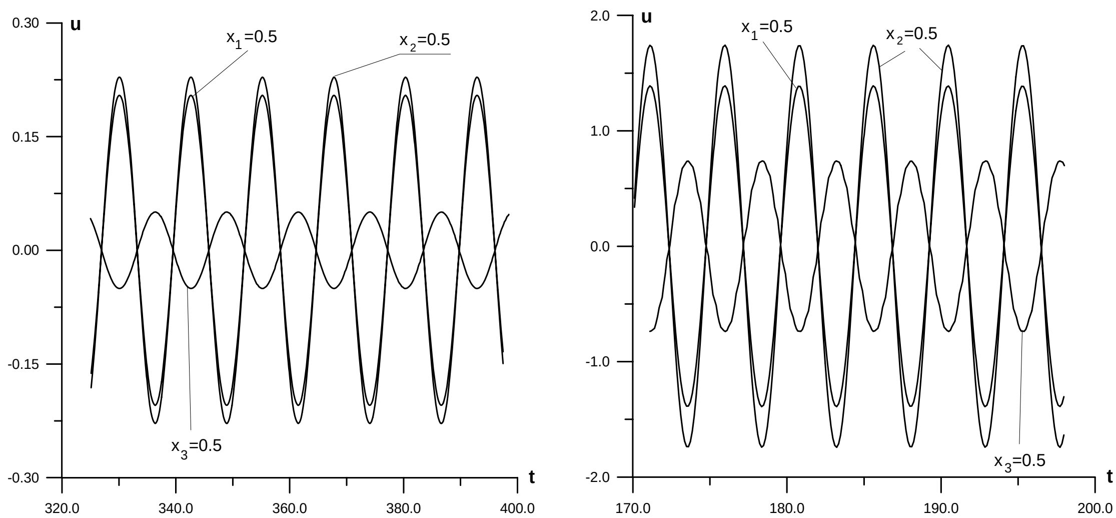

Fig. 15. Displacements in the steady state in $x_1 = x_2 = x_3 = 0.5$ for $b_{0i} = -20$ and for *a*) $p = 0.5$, *b*) $p = 1.3$

5. FINAL REMARKS

In the paper, on the example the system consisting of two rigid bodies and three noncoaxial rods, it is shown that variable cross-sections of elastic elements can be incorporated in the discussion of dynamical discrete-continuous systems longitudinally deformed. In order to use the wave approach functions describing the cross-sections ought to allow to assume the solutions of the d'Alembert type for the equations of motion. Such a type of functions are proposed in the paper. From numerical results with external force changing harmonically in time it follows that the effect of variable cross-sections is observed in the neighborhood of the resonant region. This conclusion can have a significant meaning in the studies of systems in which variable cross-sections ought to be taking into account and when the wave approach could be applied.

REFERENCES

[1] V.B.Cherepennikov, Polynomial quasisolutions of a boundary value problem for linear differential-difference equations, Functional Differential Equations, **5**, 4 (2008), 143-157.

[2] V.B.Cherepennikov, P.G.Ermolaeva, Polynomial quasisolutions of linear differential-difference equations, Opuscula Mathematica, **26**, 3 (2006), 431-443.

[3] T.P.Desmond, Theoretical and experimental investigation of stress waves in junction of three bars, J. Applied Mechanics, **48** (1981), 148-154.
[4] J.M. Gere, W.Weaver, Analysis of framed structures, New York, D.Van Nostrand Company (1965).
[5] K.F.Graff, Wave motion of elastic solids, Clarendon Press, Oxford (1975).
[6] J.Hale, Theory of functional differential equations, Springer-Verlag, New York (1977).
[7] W.Nadolski, A.Pielorz, The use of elastic waves in dynamic investigations of plane truss members, The Archive of Mechanical Engineering, **43** (1996)**,** 105-123.
[8] A.Pielorz, The investigation of the influence of a local elastic nonlinearity on vibrations of plane truss members, Engineering Transactions, **51**,1 (2003), 3-24.
[9] A.Pielorz, Nonlinear equations with a retarded argument in discrete-continuous systems, Mathematical Problems in Engineering, Vol. **2007**, Article ID 28430: 1-11, doi: 10.1155/2007/28430 (2007).
[10] A.Pielorz, Wave approach in nonlinear dynamic problems of multi-mass discrete-continuous systems, *International Journal of Pure and Applied Mathematics*, **50**, 2 (2009), 269-274.
[11] A.Pielorz, Vibrations of rod-rigid element systems with a local non-linearity, International Journal of Non-Linear Mechanics, (2010), in press.

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