Some Properties of an Indicator of a Function of Completely Regular Gamma-growth in the Half-plane

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Abstract. We consider the class $J\delta(\gamma(r))^o$ of delta-subharmonic functions of completely regular growth relatively to the function of growth γ in the half-plane. The concept of the indicator of function $v \in J\delta(\gamma(r))^o$ is entered and its some properties are studied.

Keywords: Just subharmonic function, function of growth, Fourier coefficients, indicator.

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1. Introduction

In the 60s several American authors (Rubel, Taylor [1], Miles [2], Shea, and others) started to use on large scale the Fourier series method for the study of the properties of entire and meromorphic functions. This method is efficient in the solution of several general problems of the theory of meromorphic functions and establishes its connections with Fourier series theory. One advantage of this method is its suitability for the investigation of functions of fairly irregular growth at infinity and functions of infinite order.

In the 80s important results in this direction were obtained by Kondratyuk [3], [4], [5], who generalized the Levin-Pflüger theory of entire functions of completely regular growth to meromorphic functions of arbitrary γ-type. The theory of entire functions of completely regular growth (c.r.g.) relatively to the function $r^{\rho(r)}$ ($\rho(r)$ is proximate order in sense of Valiron, $\lim\limits_{r\to\infty} \rho(r) = \rho > 0$) was created independent of each other by Levin and Pflüger in 30s. Presently it occupies prominent position in mathematics. Kondratyuk generalized a theory of entire functions of c.r.g. in two directions: 1) the growth of function was measured by the enough arbitrary function of growth γ, satisfying only to the condition

$$\gamma(2r) \le M\gamma(r) \tag{1}$$

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at some $M > 0$ and all $r > 0$; 2) were entered and considered classes of meromorphic functions of c.r.g. in a complex plane. We call a strictly positive continuous unbounded increasing function $\gamma(r)$ on $[0, \infty)$ a growth function.

In the 60s Grishin and Govorov independent of each other extended the Levin-Pflüger theory on the function of c.r.g. in a half-plane. In [6] K. G. Malyutin and Nazim Sadik were generalized the Grishin-Govorov theory to the functions of arbitrary γ-type in the upper half-plane of complex variable. As well as in works of Kondratyuk generalization of theory of Grishin-Govorov was conducted on functions growth of which is measured in relation to the function of growth satisfying a condition (1). In addition,were entered and examined the delta-subharmonic functions of c.r.g. in a half-plane.

2. Classes of functions in $\mathbb{C}_+$

In this paper we use terminology from [7] and [8]. Besides, following Titchmarsh, we will use following names and designations. If in some reasoning there is a number which not depending on the basic variables it is called as a constant. For a designation of absolute positive constants, not necessarily same, we use letters A, M, K. Can to meet the statement like "$|v(z)| < M\gamma(r)$ hence $3|v(z)| < M\gamma(r)$ "which should not cause misunderstanding.

Let $\mathbb{C}_+ = \{z : \Im z > 0\}$ be the upper half-plane. We denote by $C(a, r)$ the open disk of radius r with centre at a, and by Ω_+ the intersection of a set Ω with the half-plane $\mathbb{C}_+$: $\Omega_+ = \Omega \cap \mathbb{C}_+$. A subharmonic function v in $\mathbb{C}_+$ is said to be just subharmonic function if $\limsup_{z \to t} v(z) \le 0$ for each $t \in \mathbb{R}$. The class of just subharmonic functions in $\mathbb{C}_+$ will be denoted by JS. Let SK be the class of subharmonic functions in $\mathbb{C}_+$ possessing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$.

Functions in SK have the following properties [8]:

(a) $v(z)$ has non-tangential limits $v(t)$ almost everywhere on the real axis and $v(t) \in L^1_{\text{loc}}(-\infty, \infty)$;

(b) there exists a measure of variable sign ν on the real axis such that

$$\lim_{y \to +0} \int_a^b v(t + iy)\, dt = \nu([a, b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\})\,.$$

The measure ν is called the boundary measure of v;

(c) $d\nu(t) = v(t)\,dt + d\sigma(t)$, where σ is a singular measure with respect to Lebesgue measure.

For a function $v \in SK$, following [8] we define the corresponding full measure λ by the formula

$$\lambda(K) = 2\pi \int\limits_{\mathbb{C}_+ \cap K} \Im\zeta \, d\mu(\zeta) - \nu(K)\,,$$

where μ is the Riesz measure of v. The measure λ has the following properties:

(1) λ is the finite measure on each compact subset K of $\mathbb{C}$;

(2) λ is a positive measure outside $\mathbb{R}$;

(3) λ vanishes in the half-plane $\mathbb{C}_- = \{z : \Im z < 0\}$.

Conversely, if λ is a measure with properties (1) – (3), then there exists a function $v \in SK$ with full measure λ. The collection of properties (1) – (3) will be denoted by $\{\mathbf{G}\}$ in what follows; if, in addition, λ is also a non-negative measure in $\mathbb{R}$, then we denote the corresponding collection by $\{\mathbf{G}^+\}$.

If D is bounded subdomain of $\mathbb{C}_+$ and $D_1 = D \cup (\partial D \cap \mathbb{R})$, $v \in SK$, $z \in D$, then

$$v(z) = \frac{1}{2\pi} \iint\limits_{D_1} \frac{1}{\Im\zeta} \ln \left| \frac{z-\zeta}{z-\bar{\zeta}} \right| d\lambda(\zeta) + h(z)\,,$$

where h is a harmonic function in D, and if $[a,b] \subset \{\mathbb{R} \cap \partial D\}$, then h admits a continuous extension by zero to (a,b); we assume that $\dfrac{1}{\Im\zeta} \ln \left| \dfrac{z-\zeta}{z-\bar{\zeta}} \right|$ is extended to the real axis by continuity. The full measure λ determines a function $v \in SK$ to the same extent as the Riesz measure μ determines a subharmonic function in $\mathbb{C}$. More precisely, if v_1, $v_2 \in SK$ are two functions with full measure λ, then there exists a real entire function g such that $v_2(z) - v_1(z) = \Im g(z)$, $z \in \mathbb{C}_+$.

The following result holds [8].

Assertion 1.1. *$JS \subset SK$.*

The full measure of a function $v \in JS$ is a positive measure, which explains the term "just subharmonic function"

Let us now introduce the class of just δ-subharmonic functions $J\delta = JS - JS$.

Assertion 1.2. $J\delta = SK - SK$.

For a fixed measure λ let

$$d\lambda_k(\zeta) = \frac{\sin k\varphi}{\sin\varphi}\tau^{k-1}\, d\lambda(\zeta)\,\left(\zeta = \tau e^{i\varphi}\right), \lambda_k(r) = \lambda_k\left(\overline{C(0,r)}\right),$$

where $\frac{\sin k\varphi}{\sin\varphi}$ is defined for $\varphi = 0,\, \pi$ by continuity. In particular, $\lambda(r) = \lambda(\overline{C(0,r)})$.

The next relation is Carleman's formula in Grishin's notation:

$$\frac{1}{r^k}\int_0^\pi v(re^{i\varphi})\sin k\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\, dt + \frac{1}{r_0^k}\int_0^\pi v(r_0 e^{i\varphi})\sin k\varphi\, d\varphi\,, \tag{2}$$

in particular, for $k = 1$ we have

$$\frac{1}{r}\int_0^\pi v(re^{i\varphi})\sin\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda(t)}{t^3}\, dt + \frac{1}{r_0}\int_0^\pi v(r_0 e^{i\varphi})\sin\varphi\, d\varphi\,. \tag{3}$$

for all $r > r_0$.

Note also another inequality, which is useful in what follows:

$$|\lambda_m(r)| = \left|\iint_{\overline{C(0,r)}} d\lambda_m(\zeta)\right| = \left|\iint_{\overline{C(0,r)}} \frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}\, d\lambda(\zeta)\right| \le$$

$$m\iint_{\overline{C(0,r)}} \tau^{m-1}\, d|\lambda|(\zeta) \le mr^{m-1}|\lambda|(r)\,. \tag{4}$$

The Fourier coefficients of a function $v \in J\delta$ are defined as usual:

$$c_k(r,v) = \frac{2}{\pi}\int_0^\pi v(re^{i\theta})\sin k\theta\, d\theta, \quad k \in \mathbb{N}\,.$$

From (2) we obtain the following expressions for the Fourier coefficients for $r > r_0$:

$$c_k(r,v) = \alpha_k r^k + \frac{2r^k}{\pi}\int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\, dt, \quad k \in \mathbb{N}\,, \tag{5}$$

where $\alpha_k = r_0^{-k} c_k(r_0, v)$.

Let $v = v_+ - v_-$, let λ be the full measure of v and let $\lambda = \lambda_+ - \lambda_-$ be the Jordan decomposition of λ. We set

$$m(r,v) := \frac{1}{r}\int\limits_0^{\pi} v_+(re^{i\varphi})\sin\varphi\, d\varphi, \quad N(r,r_0,v) := N(r,v) := \int\limits_{r_0}^{r} \frac{\lambda_-(t)}{t^3}\, dt\,,$$

$$T(r,r_0,v) := T(r,v) := m(r,v) + N(r,v) + m(r_0,-v)\,,$$

where r_0 is an arbitrary positive number and $r_0 < r$; one may as well take $r_0 = 1$. In this notation Carleman's formula (3) can be written as follows:

$$T(r,v) = T(r,-v) \tag{6}$$

We now assume that the growth function γ satisfies the following condition:

$$\liminf_{r\to\infty} \frac{\gamma(r)}{r} > 0\,. \tag{7}$$

The function $v \in J\delta$ is called a function of finite γ-type if there exist constants $A, B > 0$ such that

$$T(r,v) \le \frac{A}{r}\gamma(Br), \quad r > r_0\,.$$

We denote the corresponding class of δ-subharmonic functions of finite γ-type by $J\delta(\gamma(r))$. By $JS(\gamma(r))$, we denote the class of proper subharmonic functions of finite γ-type.

Remark. If a condition (7) is not executed, we use more difficult description for the estimation of growth

$$T(r,v) := m(r,v) + N\left(r, \frac{r}{2}, v\right) + m\left(\frac{r}{2}, -v\right)\,.$$

A positive measure λ has finite γ-density if there exist positive constants A and B such that

$$N(r,\lambda) := \int\limits_{r_0}^{r} \frac{\lambda(t)}{t^3}\, dt \le \frac{A}{r}\gamma(Br)$$

for all $r > r_0$.

A positive measure λ in the complex plane is called a measure of finite γ-type if there exist positive constants A and B such that for all $r > 0$,

$$\lambda(r) \leq Ar\gamma(Br)\,. \tag{8}$$

We need the following theorem [7].

Theorem 2.1. *Let γ be a growth function and let $v \in J\delta$. Then the following two properties are equivalent:*

(i) $v \in J\delta(\gamma(r))$;

(ii) *the measure $\lambda_+(v)$ (or $\lambda_-(v)$) has finite γ-density and*

$$|c_k(r,v)| \leq A\gamma(Br), \quad k \in \mathbb{N}\,,$$

for some positive A, B and all $r > 0$.

Definition 1. *A function $v \in J\delta$ is called a delta-subharmonic function of completely regular growth relatively to $\gamma(r)$ if for all η and φ from $[0,\pi]$ there exists*

$$\lim_{r\to\infty} \frac{1}{\gamma(r)} \int\limits_{\eta}^{\varphi} v(re^{i\theta}) \sin\theta \, d\theta\,. \tag{9}$$

We denote the corresponding class of δ-subharmonic functions of c.r.g. to $\gamma(r)$ by $J\delta(\gamma(r))^o$. By $JS(\gamma(r))^o$, we denote the class of proper subharmonic functions from $J\delta(\gamma(r))^o$.

Let $\widetilde{L}^\infty[0,\pi]$ be the Banach subspace of $L^\infty[0,\pi]$ generating by the family of characteristic functions of all intercepts from $[0,\pi]$. By Cantor theorem of uniform continuity $C[0,\pi] \subset \widetilde{L}^\infty[0,\pi]$. We denote by $\mathcal{L}[0,\pi]$ any from the spaces $C[0,\pi]$, $\widetilde{L}^\infty[0,\pi]$ or $L^1[0,\pi]$. The main result of [6] is following theorem.

Theorem 2.2. *Let $v \in J\delta$. Then the following properties are equivalent:*

(i) $v \in J\delta(\gamma(r))^o$;

(ii) *$v \in J\delta(\gamma(r))$ and for all $k \in \mathbb{N}$ there exists*

$$\lim_{r\to\infty} \frac{c_k(r,v)}{\gamma(r)} = c_k\,; \tag{10}$$

(iii) *the measure* $\lambda_-(v)$ *has finite* γ*-density and for any function* ψ *from* $\mathcal{L}[0,\pi]$ *there exists*

$$\lim_{r\to\infty}\frac{1}{\gamma(r)}\int\limits_0^\pi \psi(\theta)v(re^{i\theta})\sin\theta\,d\theta\,. \tag{11}$$

Here $\lambda(v)$=$\lambda_+(v)-\lambda_-(v)$ is the full measure corresponding to the function v and $c_k(r,v)$ are the Fourier coefficients of v.

Analogous criterion for meromorphic functions in complex plane is got by Kondratyuk. We remark that if v from a class $JS(\gamma(r))^o$, then a restriction on a measure $\lambda_-(v)$ in (iii) absents ($\lambda_-(v)\equiv 0$).

3. Indicator of function $v\in J\delta(\gamma(r))^o$

In the theory of functions of c.r.g. the important role is played by concept of indicator in sense of Fragmen-Lindelöf. Other definition of indicator, based on Fourier coefficients of a function was entered by Kondratyuk. Thus its definition and definition in sense of Fragmen-Lindelöf in a case when growth function $\gamma(r)=r^\rho$ coincide. We also enter definition of the indicator delta-subharmonic functions of c.r.g. based on its Fourier coefficients.

Definition 2. *Let* $v\in J\delta(\gamma(r))^o$ *and let* c_k *be as in (10) then the function*

$$h(\theta,v)=\sum_{k=1}^{\infty}c_k\sin k\theta$$

is called the indicator of v.
Such definition is justified by that this function characterizes asymptotical behaviour of v on half-lines. Further the following lemma about peaks of Polia [9] will be useful to us.

Lemma 3.1. *Let* ψ_1, ψ_2, ψ *be positive continuous functions from* r *on* $[r_0,\infty)$ *such that the quotient* $\psi_2(r)/\psi_1(r)$ *increases and*

$$\limsup_{r\to\infty}\frac{\psi(r)}{\psi_1(r)}=\infty,\quad \limsup_{r\to\infty}\frac{\psi(r)}{\psi_2(r)}=0\,.$$

Then there exists the sequence $\{r_n\}$, $r_n\to\infty$ $(n\to\infty)$, *such that the following inequalities are valid*

$$\frac{\psi(t)}{\psi_1(t)} \le \frac{\psi(r_n)}{\psi_1(r_n)}, \quad r_0 \le t \le r_n,$$
$$\frac{\psi(t)}{\psi_2(t)} \le \frac{\psi(r_n)}{\psi_2(r_n)}, \quad r_n \le t < \infty\,.$$

Theorem 3.2. *Let the function v be from the class $J\delta(\gamma(r))^o$ then the indicator $h(\theta, v)$ belongs to $L_2[0,\pi]$.*

Proof. It follows from (1) that

$$\beta := \limsup_{r\to\infty} \frac{\ln\gamma(r)}{\ln r} < \infty\,.$$

Then $\lim_{r\to\infty}\gamma(r)/r^k = 0$ for all $k > \beta$. The inequality $|c_k(r,v)| \le A\gamma(r)$ and formula (5) yield

$$c_k(r,v) = -\frac{2r^k}{\pi}\int_r^\infty \frac{\lambda_k(t)}{t^{2k+1}}\,dt, \quad k > \beta\,. \tag{12}$$

Applying the formula of integration by parts to the integral in (12) we obtain for all $k > \beta$

$$c_k(r,v) = -\frac{1}{\pi k r^k}\iint_{C_+(0,r)} \frac{\sin k\varphi}{\Im\zeta}\tau^k\,d\lambda(\zeta) - \frac{r^k}{\pi k}\iint_{|\zeta|\geqslant r}\frac{\sin k\varphi}{\tau^k\Im\zeta}\,d\lambda(\zeta), \quad \zeta = \tau e^{i\varphi}\,. \tag{13}$$

We set $\tilde\lambda = |\lambda|$,

$$N_1(r,v) := \int_{r_0}^r \frac{\tilde\lambda(t)}{t^3}\,dt\,.$$

It follows from theorem 2.1 that the measure $\tilde\lambda$ has finite γ-density. From (13) we obtain the inequality

$$|c_k(r,v)| \le \frac{1}{\pi r^k}\int_0^r t^{k-1}\,d\tilde\lambda(t) + \frac{r^k}{\pi}\int_r^\infty \frac{d\tilde\lambda(t)}{t^{k+1}}, \quad k > \beta\,.$$

Applying the formula of integration by parts to the integral in this inequality we obtain for all $k > \beta$

$$|c_k(r,v)| \le \frac{(k+1)r^k}{\pi}\int\limits_r^\infty \frac{\tilde\lambda(t)}{t^{k+2}}\,dt - \frac{k-1}{r^k\pi}\int\limits_0^r t^{k-2}\tilde\lambda(t)\,dt =$$

$$\frac{(k+1)r^k}{\pi}\int\limits_r^\infty \frac{dN_1(t)}{t^{k-1}} - \frac{k-1}{r^k\pi}\int\limits_0^r t^{k+1}\,dN_1(t) = \tag{14}$$

$$\frac{(k^2-1)}{\pi}\left\{\int\limits_r^\infty \left(\frac{r}{t}\right)^k N_1(t)\,dt + \int\limits_0^r \left(\frac{t}{r}\right)^k N_1(t)\,dt\right\} - \frac{2k}{\pi} r N_1(r)\,.$$

Let $\limsup\limits_{r\to\infty} N_1(r)/r^{\beta-\varepsilon} = \infty$ for all $\varepsilon > 0$. By lemma 3.1 for functions $\psi(r) = N_1(r)$, $\psi_1(r) = r^{\beta-\varepsilon}$, $\psi_2(r) = r^{\beta+\varepsilon}$, we found the sequence $\{r_n\}$, $r_n \to \infty$ $(n\to\infty)$, such that

$$N_1(t) \le \left(\frac{t}{r_n}\right)^{\beta-\varepsilon}, r_0 \le t \le r_n; \quad N_1(t) \le \left(\frac{t}{r_n}\right)^{\beta+\varepsilon}, r_n \le t < \infty\,. \tag{15}$$

By (15) we obtain from (14)

$$|c_k(r_n,v)| \le \frac{2k}{\pi} N(r_n)\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2} - 1\right\} \le$$
$$\frac{Ak}{\pi}\gamma(r_n)\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2} - 1\right\}, \quad k > \beta\,.$$

This inequality yields

$$|c_k| = \lim_{r\to\infty}\frac{|c_k(r,v)|}{\gamma(r)} = \lim_{n\to\infty}\frac{|c_k(r_n,v)|}{\gamma(r_n)} \le \frac{Ak}{\pi}\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2} - 1\right\}, \quad k > \beta\,.$$

As $\varepsilon > 0$ is any number then

$$|c_k| \le \frac{Ak}{\pi}\left\{\frac{\beta^2+\beta}{k^2-\beta^2}\right\}, \quad k > \beta\,.$$

This completes the proof of Theorem 3.2.

Theorem 3.3. *If $v \in J\delta(\gamma(r))^o$ and $k \in \mathbb{N}$ then there exists finite limit*

$$\lim_{r\to\infty}\frac{1}{\gamma(r)}\int\limits_0^\pi v(re^{i\theta})\sin k\theta\,d\theta = \int\limits_0^\pi h(\theta,v)\sin k\theta\,d\theta\,.$$

This equality is derived by decomposition in Fourier series element of integration in the right part, its by member integration and limiting transition in the left part of equality.

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