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Infinitesimal Paraholomorphically Projective Transformation On Cotangent Bundle With Riemannian Extension

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ABSTRACT: The main purpose of the present paper is to study some properties of infinitesimal paraholomorphically projective transformation on T^*M with respect to the Levi-Civita connection of the Riemannian extension $(^R\nabla)$ and adapted almost paracomplex structure J. Moreover, if T^*M be admits a non-affine infinitesimal paraholomorphically projective transformation, than M and T^*M are locally flat.

Keywords: Paraholomorphically projective transformation, almost paracomplex structure, Riemannian extension, adapted frame.

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INTRODUCTION

Let M be an n −dimensional manifold and T^*M its cotangent bundle. Note that in the present paper everything will be always discussed in the C^∞- category, manifolds will be assumed to be connected and dimension $n > 1$. And let π the natural projection $T^*M \to M$. The local coordinates $\left(U, x^j\right)$, $j = 1, \ldots, n$ on M induces a system of local coordinates $\left(\pi^{-1}(U), x^j, x^{\bar{j}} = p_j\right)$, $\bar{j} = n+1, \ldots 2n$ on T^*M, where $x^{\bar{j}} = p_j$ are the components of the covector p in each cotangent space T_x^*M and $x \in U$ with respect to the natural coframe $\left\{dx^j\right\}$. We denote the set of all tensor fields of type (r,s), by $\Im_s^r(M), \Im_s^r\left(T^*(M)\right)$ on M and T^*M respectively.

The problem of determining infinitesimal holomorphically projective transformation on M and TM have been studied some authors, including (Hasegawa and Yamauchi, 1979; Hasegawa and Yamauchi, 2003; Hasegawa and Yamauchi, 2005; Tarakci et al., 2009; Gezer, 2011). Also, (Etayo and Gadea, 1992; Iscan and Magden, 2008), investigated some properties of infinitesimal paraholomorphically projective transformations on tangent bundle.

In this paper, we shall use the Levi-Civita connection of the Riemannian extension by using the horizantal and vertical lifts and we give definition and formulas almost paracomplex structure J. Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension $\left({}^R\nabla\right)$ and adapted almost paracomplex structure.

MATERIAL AND METHODS

Let ∇ be an affine connection on M. A vector field V on M is called an infinitesimal projective transformation if there exist a 1-form Ω on M such that

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X,$$

for any $X, Y \in \Im_0^1(M)$, where L_V is the Lie derivation with respect to V. In this case Ω is called the associated 1- form of V. Especially, if $\Omega = 0$ then V is called an infinitesimal affine transformation.

An almost paracomplex manifold is an almost product manifold (M,J), $J^2 = I$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of J, respectively (Cruceanu et al., 1995; Salimov et al., 2007). (M,J) be an almost paracomplex manifold with affine connection ∇. A vector field Von $M$is called an infinitesimal paraholomorphically projective transformation if there exist a 1-form Ω on M such that

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)JY + \Omega(JY)JX,$$

for any $X, Y \in \Im_0^1(M)$. In this case Ω is also called the associated 1- form of V (Prvanovic, 1971; Etayo and Gadea, 1992).

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions of a vector field X and a covector (1-form) field ω on M, respectively. According to the induced coordinates the vertical lift ${}^V\omega$ of ω, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are obtained as follows

$${}^V\omega = \omega_i \partial_{\bar{i}}, \tag{1}$$

$${}^HX = X^i\partial_i + p_h\Gamma_{ij}^h X^j \partial_{\bar{i}}, \tag{2}$$

$${}^CX = X^j\partial_i - p_h\partial_i X^h \partial_{\bar{i}},$$

where $\partial_i = \frac{\partial}{\partial x^i}, \partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$ and Γ_{ij}^h are the coefficients of symmetric (torsion-free) affine connection ∇ on M (Yano and Ishihara, 1973). For arbitrary $X, Y \in \Im_0^1(M)$ and $\theta, \omega \in \Im_1^0(M)$, the Lie bracket operation of vertical and horizontal vector fields on T^*M is given as follows

$$\begin{cases} [{}^H X, {}^H Y] = {}^H [X, Y] + {}^V\big(p \circ R(X, Y)\big) \\ [{}^H X, {}^V \omega] = {}^V(\nabla_X \omega) \\ [{}^V \theta, {}^V \omega] = 0, \end{cases} \tag{3}$$

where $R = R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature tensor of the symmetric connection ∇ (Yano and Ishihara, 1973).

The adapted frame

The adapted frame $\{E_\alpha\} = \{E_j, E_{\bar{j}}\}$ on each induced coordinate neighbourhood $\pi^{-1}(U)$ of T^*M is given by (Yano and Ishihara, 1973)

$$\left.\begin{aligned} E_j &= {}^H X_{(j)} = \partial_j + p_a \Gamma_{hj}^a \partial_{\bar{h}}, \\ E_{\bar{j}} &= {}^V \theta_{(j)} = \partial_{\bar{j}}, \end{aligned}\right\} \tag{4}$$

where

$$X_{(j)} = \frac{\partial}{\partial x^j}, \theta^j = dx^j, j = 1, \ldots, n,$$

the indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 2n$ denote the indices according to the adapted frame. It follows from (1), (2) and (4) that

$${}^V\omega = \begin{pmatrix} 0 \\ \omega_j \end{pmatrix}$$

$${}^H X = \begin{pmatrix} X^j \\ 0 \end{pmatrix}$$

according to the adapted frame $\{E_\alpha\}$.

Lemma 1 The Lie bracket of the adapted frame of T^*M satisfies the following identities (Yano and Ishihara, 1973)

$$[E_i, E_j] = p_s R_{ijl}^s E_{\bar{l}},$$
$$[E_i, E_{\bar{j}}] = -\Gamma_{il}^j E_{\bar{l}},$$
$$[E_{\bar{i}}, E_{\bar{j}}] = 0,$$

where $R_{ijl}^s = \partial_i \Gamma_{jl}^s - \partial_j \Gamma_{il}^s + \Gamma_{ik}^s \Gamma_{jl}^k - \Gamma_{jk}^s \Gamma_{il}^k$ indicates the Riemannian curvature tensor of (M, g).

Lemma 2 Let V be a vector field of T^*M with the components $(v^h, v^{\bar{h}})$. Then, the Lie derivatives of the adapted frame and the dual basis are obtained as follows (Bilen, 2019):

1. $L_V E_i = -(E_i v^k) E_k - \left(v^a p_s R_{iak}{}^s + E_i v^{\bar{k}} - v^{\bar{a}} \Gamma_{ik}^a\right) E_{\bar{k}}$,
2. $L_V E_{\bar{i}} = -(E_{\bar{i}} v^k) E_k - \left(v^a \Gamma_{ak}^i + E_{\bar{i}} v^{\bar{k}}\right) E_{\bar{k}},$
3. $L_V dx^h = (E_k v^h) dx^k + (E_{\bar{k}} v^h) \delta p_k,$
4. $L_V \delta p_h = (v^a p_s R_{kah}{}^s - v^{\bar{a}} \Gamma_{kh}^a + (E_k v^{\bar{m}}) \delta_h^m) dx^k + (v^a \Gamma_{ah}^k + (E_{\bar{k}} v^{\bar{m}}) \delta_h^m) \delta p_k.$

{For more work on tangent bundles see (Hasegawa and Yamauchi, 2003; Gezer, 2011)}.

Riemannian Extension

A pseudo-Riemannian metric ${}^{R}\nabla \in \Im_2^0(T^*M)$ is given by (Yano and Ishihara, 1973).

$$ {}^{R}\nabla({}^{C}X, {}^{C}Y) = -\gamma(\nabla_X Y + \nabla_Y X), $$

for any $X, Y \in \Im_0^1(M)$, where

$$ -\gamma(\nabla_X Y + \nabla_Y X) = p_m\left(X^j \nabla_j Y^m + Y^j \nabla_j X^m\right), $$

${}^{R}\nabla \in \Im_2^0(T^*M)$ with the following components in $\pi^{-1}(U)$

$$ {}^{R}\nabla = \left({}^{R}\nabla_{JI}\right) = \begin{pmatrix} -2p_h\Gamma_{ji}^h & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix} $$

relative to the natural frame, where δ_j^i is the Kronecker delta. The indices $i, j, k, \ldots = 1, \ldots, 2n$ correspond to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$. The analyzed tensor field defines a pseudo-Riemannian metric in T^*M and a line element of the pseudo-Riemannian metric ${}^{R}\nabla$ is given by the formula

$$ ds^2 = 2dx^i \delta p_i, $$

where

$$ \delta p_i = dp_i - p_h \Gamma_{ji}^h dx^i. $$

This metric is called the Riemannian extension of the symmetric affine connection ∇ (Patterson and Walker, 1952; Yano and Ishihara, 1973). Any tensor field of type $(0,2)$ is entirely detected by its action of ${}^{H}X$ and ${}^{V}\omega$ on T^*M (Yano and Ishihara, 1973). Then the Riemannian extension ${}^{R}\nabla$ is defined by

$$ {}^{R}\nabla({}^{V}\omega, {}^{V}\theta) = 0, $$

$$ {}^{R}\nabla({}^{V}\omega, {}^{H}X) = {}^{V}\left(\omega(X)\right) = \left(\omega(X)\right) \circ \pi, $$

$$ {}^{R}\nabla({}^{H}X, {}^{H}Y) = 0 $$

for any $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$ (Aslanci et al., 2010).

The Levi-Civita connection of ${}^{R}\nabla$

${}^{C}\nabla$ is the Levi-Civita connection of ${}^{R}\nabla$, because of ${}^{C}\nabla({}^{R}\nabla) = 0$. (${}^{C}\nabla$ is called the complete lift of ∇ to T^*M) The Levi-Civita connection of ${}^{C}\nabla$ in $\pi^{-1}(U) \subset T^*M$ are given by

$$ {}^{C}\Gamma_{ji}^h = \Gamma_{ji}^h $$

$$ {}^{C}\Gamma_{j\bar{i}}^{\bar{h}} = -\Gamma_{jh}^i $$

$$ {}^{C}\Gamma_{ji}^{\bar{h}} = \frac{1}{2} p_m\left(R_{jih}^m - R_{ihj}^m + R_{hji}^m\right) = p_m R_{hij}^m $$

$$ {}^{C}\Gamma_{\bar{j}i}^{h} = {}^{C}\Gamma_{j\bar{i}}^{h} = {}^{C}\Gamma_{\bar{j}\bar{i}}^{h} = {}^{C}\Gamma_{\bar{j}\bar{i}}^{\bar{h}} = {}^{C}\Gamma_{\bar{j}i}^{\bar{h}} = 0 $$

with respect to adapted frame $\{E_\alpha\}$, where Γ_{ji}^h denote the Christoffel symbols constructed with g_{ji} on M (Aslanci et al, 2010).

Let us consider a tensor field J of type $(1,1)$ on T^*M defined by

$$ J^{H}X = -{}^{H}X, J^{V}\omega = {}^{V}\omega, $$

for any $X \in \Im_0^1(M)$, i.e., $JE_i = -E_i$, $JE_{\bar{i}} = E_{\bar{i}}$. Then we obtain $J^2 = I$. Therefore J is an almost paracomplex structure on T^*M. This almost paracomplex structure is called adapted almost paracomplex structure (Etayo and Gadea, 1992).

RESULTS AND DISCUSSION

Theorem 3 Let (M,g) be a Riemannian manifold and T^*M be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vector field V is an infinitesimal paraholomorphically projective transformation with associated 1- form Ω on T^*M if and only if there exist $B=(B^h)\in\Im_0^1(M), D=(D_h)\in\Im_1^0(M)$ and $A=\left(A_i^h\right), C=\left(C_i^h\right)\in\Im_1^1(M)$ satisfying

1. $\begin{pmatrix} v^k \\ v^{\bar{k}} \end{pmatrix}=\begin{pmatrix} p^s A_s^k+B^k \\ D_k+p_a C_k^a+4\varphi p_k+2p_a p_k \Psi^a \end{pmatrix}$
2. $\nabla_j A^{ki}=0,\ \nabla_j C_i^i=0$
3. $\nabla_j\varphi=0,\ \nabla_j\psi=0,\ \nabla_j\Psi^i=0$
4. $A^{ia}R_{aij}^s=0$
5. $A_s^a R_{aij}^k+A_h^k R_{sij}^h=0$
6. $\nabla_i\nabla_j B^k+B^a R_{aij}^k=2\Omega_i\delta_j^k+2\Omega_j\delta_i^k=L_B\Gamma_{ij}^k$
7. $\nabla_i R_{jak}^s-\nabla_a R_{kji}^s=0$
8. $R_{jhi}^s\Psi^h=0$
9. $\nabla_i\nabla_j D_k+D_a R_{kji}^a=0$
10. $C_k^h R_{jih}^s+C_a^s R_{ijk}^a=0$
11. $\Omega_j=\frac{1}{4n}\nabla_i\nabla_j B^j,\ \Omega_{\bar{j}}=\Psi^j$

where $V=\begin{pmatrix} v^k \\ v^{\bar{k}} \end{pmatrix}=v^k E_k+v^{\bar{k}}E_{\bar{k}},\ \Omega=\left(\Omega_j dx^j+\Omega_{\bar{j}}\delta y^j\right)$.

Proof. Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let V be an infinitesimal paraholomorphically projective transformation with the associated 1- form Ω on T^*M

$$(L_V\nabla)(X,Y)=\Omega(X)Y+\Omega(Y)X+\Omega(JX)JY+\Omega(JY)JX$$

for any $X,Y\in\Im_0^1(M)$.
From

$$\left(L_V{}^R\nabla\right)\left(E_{\bar{i}},E_{\bar{j}}\right)=\Omega\left(E_{\bar{i}}\right)E_{\bar{j}}+\Omega\left(E_{\bar{j}}\right)E_{\bar{i}}+\Omega\left(JE_{\bar{i}}\right)JE_{\bar{j}}+\Omega\left(JE_{\bar{j}}\right)JE_{\bar{i}}$$

we obtain

$$\left(L_V{}^R\nabla\right)\left(E_{\bar{i}},E_{\bar{j}}\right)=2\left(\Omega_{\bar{i}}\delta_k^j+\Omega_{\bar{j}}\delta_k^i\right)E_{\bar{k}} \tag{5}$$

also

$$\left(L_V{}^R\nabla\right)\left(E_{\bar{i}},E_{\bar{j}}\right)=\left[\partial_{\bar{i}}\left(\partial_{\bar{j}}v^k\right)\right]E_k+\left[\partial_{\bar{i}}\left(\partial_{\bar{j}}v^{\bar{k}}\right)\right]E_{\bar{k}} \tag{6}$$

from (5) and (6) we obtain

$$\partial_{\bar{i}}\left(\partial_{\bar{j}}v^k\right)=0\Rightarrow v^k=p^s A_s^k+B^k \tag{7}$$

and

$$\partial_{\bar{i}}\left(\partial_{\bar{j}}v^{\bar{k}}\right)=2\left(\Omega_{\bar{i}}\delta_k^j+\Omega_{\bar{j}}\delta_k^i\right). \tag{8}$$

Contracting k and j in (8), we have

$$\Omega_{\bar{i}}=\partial_{\bar{i}}\psi, \tag{9}$$

where $\psi = \frac{1}{2n+2}\partial_{\bar{j}}v^{\bar{j}}$. If we use the expression (9) in (8), expression (8) is rewritten as follows:

$$\partial_{\bar{i}}\left(\partial_{\bar{j}}v^{\bar{k}}\right) = 2\left(\partial_{\bar{i}}\psi\right)\delta_k^j + 2\left(\partial_{\bar{j}}\psi\right)\delta_k^i. \tag{10}$$

Differentiating (10) partially, we have

$$\begin{aligned}\partial_{\bar{h}}\partial_{\bar{i}}\partial_{\bar{j}}v^{\bar{k}} &= 2\partial_{\bar{h}}\partial_{\bar{i}}\psi\delta_k^j + 2\partial_{\bar{h}}\partial_{\bar{j}}\psi\delta_k^i \\ &= 2\partial_{\bar{h}}\partial_{\bar{i}}\psi\delta_k^j + 2\partial_{\bar{h}}\partial_{\bar{i}}\psi\delta_k^i\delta_i^j \\ &= \partial_{\bar{h}}\partial_{\bar{i}}\left(4\psi\delta_k^j\right)\end{aligned}$$

from here we get

$$\partial_{\bar{h}}\partial_{\bar{i}}\left(\partial_{\bar{j}}v^{\bar{k}} - 4\psi\delta_k^j\right) = 0.$$

Written here as

$$M_k^{ij} = \partial_{\bar{i}}\left(\partial_{\bar{j}}v^{\bar{k}} - 4\psi\delta_k^j\right) \tag{11}$$

and

$$C_k^j + p_a M_k^{aj} = \partial_{\bar{j}}v^{\bar{k}} - 4\psi\delta_k^j, \tag{12}$$

where C_k^j and M_k^{ij} are certain functions which depend only on the variables (x^h). Also

$$M_k^{ij} + M_k^{ji} = \partial_{\bar{i}}\partial_{\bar{j}}v^{\bar{k}} - 4\partial_{\bar{i}}\psi\delta_k^j + \partial_{\bar{j}}\partial_{\bar{i}}v^{\bar{k}} - 4\partial_{\bar{j}}\psi\delta_k^i.$$

Using (10) in above equation

$$M_k^{ij} = \frac{1}{2}\left(M_k^{ij} - M_k^{ji}\right) = 2\left[\left(\partial_{\bar{j}}\psi\right)\delta_k^i - \left(\partial_{\bar{i}}\psi\right)\delta_k^j\right]. \tag{13}$$

Contracting k and j in (12), we have

$$C_k^k + p_a M_k^{ak} = (2-2n)\psi.$$

From which

$$\psi = \frac{1}{2-2n}C_k^k + p_a\frac{1}{2-2n}M_k^{ak}$$

and we get

$$\psi = \varphi + p_a\Psi^a, \tag{14}$$

where $\varphi = \frac{1}{2-2n}C_k^k$ and $\Psi^a = \frac{1}{2-2n}M_k^{ak}$, from which we have

$$\Omega_{\bar{i}} = \partial_{\bar{i}}\psi = \Psi^i. \tag{15}$$

If used (13) and (14) in (12) we get

$$\partial_{\bar{j}}v^{\bar{k}} = C_k^j + 4\varphi\delta_k^j + 2p_a\Psi^a\delta_k^j + 2p_k\Psi^j$$

and

$$v^{\bar{k}} = D_k + p_a C_k^a + 4\varphi p_k + 2p_a p_k\Psi^a, \tag{16}$$

where D_k are certain functions which depend only on (x^h). The coordinat transformation rule implies that $D = (D_k) \in \mathfrak{I}_1^0(M)$.

Next, from

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)JY + \Omega(JY)JX,$$

we have

$$\left(L_V{}^R\nabla\right)\left(E_{\bar{i}}, E_j\right) = 0$$

or

$$\left(L_V{}^R\nabla\right)\left(E_i, E_{\bar{j}}\right) = 0$$

from which, we get

$$0 = [\nabla_j A^{ki}]E_k + [A^{ia}p_s R^s_{kaj} + v^a R^i_{jak} + \nabla_j C^i_k + 2p_k(\nabla_j \Psi^i) + 2p_a \delta^i_k \nabla_j \Psi^a + 4(\partial_j \varphi)\delta^i_k]E_{\bar{k}}.$$

Therefore,

$$\nabla_j A^{ki} = 0 \tag{17}$$

and

$$A^{ia}p_s R^s_{kaj} + v^a R^i_{jak} + \nabla_j C^i_k + 2p_s(\nabla_j \Psi^s) + 2p_a \delta^i_k \nabla_j \Psi^a + 4(\partial_j \varphi)\delta^i_k = 0. \tag{18}$$

Contracting k and i in (18), we have

$$\begin{cases} \nabla_j C^i_i = 0, \\ \nabla_j \varphi = 0 \end{cases} \tag{19}$$

and

$$\nabla_j \Psi^s = \frac{1}{2(n+1)} A^{ia} R^s_{aij}. \tag{20}$$

Lastly, from

$$\left(L_V{}^R\nabla\right)\left(E_i, E_j\right) = \left(2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i\right)E_k$$

we obtain

$$\begin{aligned}\left(2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i\right)E_k &= [\nabla_i \nabla_j v^k + v^a R^k_{aij} + A^{hk}p_s R^s_{hji}]E_k \\ &+ \Big[p_s\Big((\nabla_i v^h)R^s_{khj} + (\nabla_j v^h)R^s_{khi} - \left(E_{\bar{h}} v^{\bar{k}}\right)R^s_{hji}\Big) \\ &+ v^a p_s(\nabla_i R^s_{jak} - \nabla_a R^s_{kji}) + \left(v^{\bar{a}} R^a_{kji} + \nabla_i \nabla_j v^{\bar{k}}\right)\Big]E_{\bar{k}}\end{aligned}$$

from which, using (7) and (16), we obtain

$$\nabla_i \nabla_j A^k_s + A^a_s R^k_{aij} + A^k_h R^h_{sij} = 0, \tag{21}$$

$$\nabla_i \nabla_j B^k + B^a R^k_{aij} = 2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i = L_B \Gamma^k_{ij}, \tag{22}$$

$$\nabla_i R^s_{jak} - \nabla_a R^s_{kji} = 0, \tag{23}$$

$$\nabla_i \nabla_j \Psi^s + R^s_{jhi} \Psi^h = 0, \tag{24}$$

$$\nabla_i \nabla_j D_k + D_a R^a_{kji} = 0, \tag{25}$$

$$(\nabla_i B^a)R^s_{kaj} + (\nabla_j B^a)R^s_{kai} + C^h_k R^s_{jhi} + C^s_a R^a_{kji} + \nabla_i \nabla_j C^s_k = 0. \tag{26}$$

From (26), we get

$$K_{ij} = (\nabla_i B^a)R^s_{kaj} + (\nabla_j B^a)R^s_{kai} + C^h_k R^s_{jhi} + C^s_a R^a_{kji} + \nabla_i \nabla_j C^s_k = 0,$$

$$K_{ji} = (\nabla_j B^a)R^s_{kai} + (\nabla_i B^a)R^s_{kaj} + C^h_k R^s_{ihj} + C^s_a R^a_{kij} + \nabla_j \nabla_i C^s_k = 0.$$

$$K_{ij} - K_{ji} = C^h_k(R^s_{jhi} - R^s_{ihj}) + C^s_a(R^a_{kji} - R^a_{kij}) = 0$$

$$C^h_k R^s_{jih} + C^s_a R^a_{ijk} = 0 \tag{27}$$

Contracting j and k in (22), we obtain

$$\Omega_i = \frac{1}{4n}\nabla_i \nabla_j B^j. \tag{28}$$

This completes the proof.

Theorem 4 Let (M, g) be a Riemannian manifold and T^*M be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. If T^*M admits a non-affine infinitesimal paraholomorphically projective transformation, than M and T^*M are locally flat.

Proof. Let V be non-affine infinitesimal paraholomorphically projective transformation on T^*M, using (3) in the expression of theorem 3, we have $\nabla_i\|\Psi\|^2 = \nabla_j\|\partial\psi\|^2 = 0$. Hence, $\|\Psi\|$ and $\|\partial\psi\|$ are constant on M. Suppose that M is non-locally flat, then $\Psi = \partial\psi = 0$ by virtue of (9) and (3) in the expression of theorem 3, that is, V is an infinitesimal affine transformation. This is a contradiction. Therefore, M is locally flat. In this case T^*M is locally flat.

Corollary 5 Let (M, g) be a Riemannian manifold and T^*M be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vertical vector field V is an infinitesimal paraholomorphically projective transformation with associated 1- form Ω on T^*M if and only if there exist $D = (D_h) \in \Im_1^0(M)$ and $C = \left(C_i^h\right) \in \Im_1^1(M)$ satisfying

1. $\begin{pmatrix} v^k \\ v^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ D_k + p_a C_k^a + 4\varphi p_k + 2p_a p_k \Psi^a \end{pmatrix}$
2. $\nabla_j C_k^i = 0$
3. $\nabla_j\varphi = 0,\ \nabla_j\psi = 0,\ \nabla_j\Psi^i = 0$
4. $\nabla_i\nabla_j D_k + D_a R_{kji}^a = 0$
5. $C_a^s R_{kji}^a + C_k^h R_{jhi}^s = 0$
6. $\Psi^h R_{jhi}^s = 0$
7. $\Psi^a R_{jki}^s + \Psi^s R_{kji}^a = 0$
8. $\Omega_j = 0,\ \Omega_{\bar{j}} = \Psi^j$

where $V = \begin{pmatrix} 0 \\ v^{\bar{k}} \end{pmatrix} = v^{\bar{k}} E_{\bar{k}},\ \Omega = \left(\Omega_j dx^j + \Omega_{\bar{j}}\delta y^j\right)$.

CONCLUSION

In this article, we use the Levi-Civita connection of the Riemannian extension and we give definition and formulas almost paracomplex structure J. Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension $({}^R\nabla)$ and adapted almost paracomplex structure J.

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