Infinitesimal Paraholomorphically Projective Transformation On Cotangent Bundle With Riemannian Extension

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ABSTRACT: The main purpose of the present paper is to study some properties of infinitesimal paraholomorphically projective transformation on $T^*M$ with respect to the Levi-Civita connection of the Riemannian extension $(\nabla^R)$ and adapted almost paracomplex structure $J$. Moreover, if $T^*M$ be admits a non-affine infinitesimal paraholomorphically projective transformation, than $M$ and $T^*M$ are locally flat.

Keywords: Paraholomorphically projective transformation, almost paracomplex structure, Riemannian extension, adapted frame.

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INTRODUCTION

Let \( M \) be an \( n \)-dimensional manifold and \( T^*M \) its cotangent bundle. Note that in the present paper everything will be always discussed in the \( C^\infty \)-category, manifolds will be assumed to be connected and dimension \( n > 1 \). And let \( \pi \) the natural projection \( T^*M \rightarrow M \). The local coordinates \((U, x^j)\), \( j = 1, \ldots, n \) on \( M \) induces a system of local coordinates \((\pi^{-1}(U), x^j, x^\bar{j} = p_j)\), \( \bar{j} = n + 1, \ldots, 2n \) on \( T^*M \), where \( x^\bar{j} = p_j \) are the components of the covector \( p \) in each cotangent space \( T^*_xM \) and \( x \in U \) with respect to the natural coframe \( \{dx^j\} \). We denote the set of all tensor fields of type \((r, s)\), by \( \mathfrak{S}_r^s(M), \mathfrak{S}_r^s(T^*(M)) \) on \( M \) and \( T^*M \) respectively.

The problem of determining infinitesimal holomorphically projective transformation on \( M \) and \( TM \) have been studied some authors, including (Hasegawa and Yamauchi, 1979; Hasegawa and Yamauchi, 2003; Hasegawa and Yamauchi, 2005; Tarakci et al., 2009; Gezer, 2011). Also, (Etayo and Gadea, 1992; Iscan and Magden, 2008), investigated some properties of infinitesimal paraholomorphically projective transformations on tangent bundle.

In this paper, we shall use the Levi-Civita connection of the Riemannian extension by using the horizontal and vertical lifts and we give definition and formulas almost paracomplex structure \( J \). Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension \((^R\nabla)\) and adapted almost paracomplex structure.

MATERIAL AND METHODS

Let \( \nabla \) be an affine connection on \( M \). A vector field \( V \) on \( M \) is called an infinitesimal projective transformation if there exist a 1-form \( \Omega \) on \( M \) such that

\[
(L_\nabla \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,
\]

for any \( X, Y \in \mathfrak{g}(\mathfrak{m}, \mathfrak{n}) \), where \( L_\nabla \) is the Lie derivation with respect to \( V \). In this case \( \Omega \) is called the associated 1-form of \( V \). Especially, if \( \Omega = 0 \) then \( V \) is called an infinitesimal affine transformation.

An almost paracomplex manifold is an almost product manifold \((M,J), J^2 = I\), such that the two eigenbundles \( T^+M \) and \( T^-M \) associated to the two eigenvalues \(+1 \) and \(-1\) of \( J \), respectively (Cruceanu et al., 1995; Salimov et al., 2007). \((M,J)\) be an almost paracomplex manifold with affine connection \( \nabla \). A vector field \( V \) on \( M \) is called an infinitesimal paraholomorphically projective transformation if there exist a 1-form \( \Omega \) on \( M \) such that

\[
(L_\nabla \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)Y + \Omega(JY)JX,
\]

for any \( X, Y \in \mathfrak{g}(\mathfrak{m}, \mathfrak{n}) \). In this case \( \Omega \) is also called the associated 1-form of \( V \) (Prvanovic, 1971; Etayo and Gadea, 1992).

Let \( X = X^i \frac{\partial}{\partial x^i} \) and \( \omega = \omega^i dx^i \) be the local expressions of a vector field \( X \) and a covector (1-form) field \( \omega \) on \( M \), respectively. According to the induced coordinates the vertical lift \( ^V\omega \) of \( \omega \), the horizontal lift \( ^H\omega \) and the complete lift \( ^C\omega \) of \( \omega \) are obtained as follows

\[
\begin{align*}
^V\omega &= \omega^i \partial_i, \\
^H\omega &= X^i \partial_i + p_i^h X^i \partial^h, \\
^C\omega &= X^i \partial_i - p_i^h \partial_i X^h \partial^h.
\end{align*}
\]
where \(\partial_i = \frac{\partial}{\partial x^i}, \partial_i = \frac{\partial}{\partial x^i} \) and \(\Gamma^h_{ij}\) are the coefficients of symmetric (torsion-free) affine connection \(\nabla\) on \(M\) (Yano and Ishihara, 1973). For arbitrary \(X, Y \in \mathfrak{X}_0(M)\) and \(\theta, \omega \in \Omega^1(M)\), the Lie bracket operation of vertical and horizontal vector fields on \(T^*M\) is given as follows

\[
\begin{align*}
\{^H X, ^H Y\} &= \nabla [X, Y] + \nabla (p \circ R(X, Y)) \\
\{^H X, ^V \omega\} &= \nabla (\nabla X \omega) \\
\{^V \theta, ^V \omega\} &= 0,
\end{align*}
\]

where \(R = R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\) is the curvature tensor of the symmetric connection \(\nabla\) (Yano and Ishihara, 1973).

**The adapted frame**

The adapted frame \(\{E_\alpha\} = \{E_j, E_\bar{j}\}\) on each induced coordinate neighbourhood \(\pi^{-1}(U)\) of \(T^*M\) is given by (Yano and Ishihara, 1973)

\[
\begin{align*}
E_j &= ^H X(j) = \partial_j + p_{a \bar{j}} \Gamma^a_{hj} \partial_h, \\
E_\bar{j} &= ^V \theta(j) = \partial_\bar{j},
\end{align*}
\]

where

\[
X(j) = \frac{\partial}{\partial x^j}, \theta^j = dx^j, j = 1, \ldots, n,
\]

the indices \(\alpha, \beta, \gamma, \ldots = 1, \ldots, 2n\) denote the indices according to the adapted frame. It follows from (1), (2) and (4) that

\[
\begin{align*}
^V \omega &= \begin{pmatrix} 0 \\ \omega_j \end{pmatrix} \\
^H X &= \begin{pmatrix} \bar{x}^j \\ 0 \end{pmatrix}
\end{align*}
\]

according to the adapted frame \(\{E_\alpha\}\).

**Lemma 1** The Lie bracket of the adapted frame of \(T^*M\) satisfies the following identities (Yano and Ishihara, 1973)

\[
\begin{align*}
[E_i, E_j] &= p_s R^s_{ijkl} E_l, \\
[E_i, E_\bar{j}] &= -\Gamma^j_{il} E_l, \\
[E_\bar{i}, E_\bar{j}] &= 0,
\end{align*}
\]

where \(R^s_{ijkl} = \partial_i \Gamma^s_{jk} - \partial_j \Gamma^s_{il} + \Gamma^s_{ik} \Gamma^s_{jl} - \Gamma^s_{jl} \Gamma^s_{ik}\) indicates the Riemannian curvature tensor of \((M, g)\).

**Lemma 2** Let \(V\) be a vector field of \(T^*M\) with the components \(\left(v^h, v^\bar{h}\right)\). Then, the Lie derivatives of the adapted frame and the dual basis are obtained as follows (Bilen, 2019):

1. \(L_V E_i = -(E_i v^h) E_k - \left(v^a p_s R^s_{iakh} + E_i v^{\bar{h}} - v^a \Gamma^a_{ih} \right) E_k\).
2. \(L_V E_\bar{i} = -(E_\bar{i} v^h) E_k - \left(v^a \Gamma^a_{ih} + E_\bar{i} v^{\bar{h}} \right) E_k\).
3. \(L_V dx^k = \left(E_k v^h\right) dx^k + \left(E_k v^{\bar{h}}\right) \delta p_k\).
4. \(L_V \delta p_k = \left(v^a p_s R^s_{kah} - v^a \Gamma^a_{kh} + \left(E_k v^m\right) \delta^m_h \right) dx^k + \left(v^a \Gamma^a_{kh} + \left(E_k v^m\right) \delta^m_h \right) \delta p_k\).

{For more work on tangent bundles see (Hasegawa and Yamauchi, 2003; Gezer, 2011).}
Riemannian Extension

A pseudo-Riemannian metric \( \overset{\text{R}}{\nabla} \in \mathcal{S}_0^0(T^*M) \) is given by (Yano and Ishihara, 1973).

\[
\overset{\text{R}}{\nabla}(\,^C X, \,^C Y) = -\gamma(\nabla_X Y + \nabla_Y X),
\]

for any \( X,Y \in \mathcal{S}_0^1(M) \), where

\[
-\gamma(\nabla_X Y + \nabla_Y X) = p_m \left( X^j \nabla_j Y^m + Y^j \nabla_j X^m \right),
\]

\( \overset{\text{R}}{\nabla} \in \mathcal{S}_0^0(T^*M) \) with the following components in \( \pi^{-1}(U) \)

\[
\overset{\text{R}}{\nabla} = \left( \overset{\text{R}}{\nabla}_{ij} \right) = \begin{pmatrix} -2p_i \Gamma^h_{ji} & \delta^i_j \\ \delta^i_j & 0 \end{pmatrix}
\]

relative to the natural frame, where \( \delta^i_j \) is the Kronecker delta. The indices \( i,j,k,\ldots = 1,\ldots,2n \) correspond to the natural frame \( \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\} \). The analyzed tensor field defines a pseudo-Riemannian metric in \( T^*M \) and a line element of the pseudo-Riemannian metric \( \overset{\text{R}}{\nabla} \) is given by the formula

\[
ds^2 = 2dx^i \delta p_i,
\]

where

\[
\delta p_i = dp_i - p_h \Gamma^h_{ji} dx^i.
\]

This metric is called the Riemannian extension of the symmetric affine connection \( \nabla \) (Patterson and Walker, 1952; Yano and Ishihara, 1973). Any tensor field of type \((0,2)\) is entirely detected by its action of \( ^H X \) and \( ^V \omega \) on \( T^*M \) (Yano and Ishihara, 1973). Then the Riemannian extension \( \overset{\text{R}}{\nabla} \) is defined by

\[
\overset{\text{R}}{\nabla}(^V \omega, ^V \theta) = 0,
\]

\[
\overset{\text{R}}{\nabla}(^V \omega, ^H X) = (^V \omega(X)) \circ \pi,
\]

\[
\overset{\text{R}}{\nabla}(^H X, ^H Y) = 0
\]

for any \( X,Y \in \mathcal{S}_0^1(M) \) and \( \omega, \theta \in \mathcal{S}_0^0(M) \) (Aslanci et al., 2010).

The Levi-Civita connection of \( \overset{\text{R}}{\nabla} \)

\( \overset{\text{C}}{\nabla} \) is the Levi-Civita connection of \( \overset{\text{R}}{\nabla} \), because of \( \overset{\text{C}}{\nabla}(\overset{\text{R}}{\nabla}) = 0 \). \( \overset{\text{C}}{\nabla} \) is called the complete lift of \( \nabla \) to \( T^*M \) (Aslanci et al., 2010). The Levi-Civita connection of \( \overset{\text{C}}{\nabla} \) in \( \pi^{-1}(U) \subset T^*M \) are given by

\[
\overset{\text{C}}{\Gamma}^h_{ji} = \Gamma^h_{ji}
\]

\[
\overset{\text{C}}{\Gamma}^h_{ji} = -\Gamma^i_{jh}
\]

\[
\overset{\text{C}}{\Gamma}^h_{ji} = \frac{1}{2} p_m \left( R^m_{jih} - R^m_{ihj} + R^m_{hij} \right) = p_m R^m_{hij}
\]

\[
\overset{\text{C}}{\Gamma}^h_{ji} = \overset{\text{C}}{\Gamma}^h_{ji} = \overset{\text{C}}{\Gamma}^h_{ji} = \overset{\text{C}}{\Gamma}^h_{ji} = 0
\]

with respect to adapted frame \( \{ E_\alpha \} \), where \( \Gamma^h_{ji} \) denote the Christoffel symbols constructed with \( g_{jl} \) on \( M \) (Aslanci et al., 2010).

Let us consider a tensor field \( J \) of type \((1,1)\) on \( T^*M \) defined by

\[
J^H X = -^H X,^V \omega = ^V \omega,
\]

for any \( X \in \mathcal{S}_0^1(M) \), i.e., \( JE_i = -E_i, JE_i = E_i \). Then we obtain \( J^2 = I \). Therefore \( J \) is an almost paracomplex structure on \( T^*M \). This almost paracomplex structure is called adapted almost paracomplex structure (Etayo and Gadea, 1992).
RESULTS AND DISCUSSION

**Theorem 3** Let \((M, g)\) be a Riemannian manifold and \(T^*M\) be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vector field \(V\) is an infinitesimal paraholomorphically projective transformation with associated 1-form \(\Omega\) on \(T^*M\) if and only if there exist 

\[ B = (B^h) \in \mathfrak{S}_0^1(M), D = (D_h) \in \mathfrak{S}_1^0(M) \text{ and } A = (A^i_h), C = (C^h_i) \in \mathfrak{S}_1^1(M) \]

satisfying

1. \( (v^k_b) = \left( D_k + p_a C^a_i k + 4 \phi p_k + 2 p_a p_k \psi^a \right) \)
2. \( \nabla_j A^{ki} = 0, \nabla_j C^i_j = 0 \)
3. \( \nabla_j \phi = 0, \nabla_j \psi = 0, \nabla_j \psi^i = 0 \)
4. \( A^{ia} R^{a}_{bij} = 0 \)
5. \( A^a R^{k}_{alj} + A^k R^{h}_{sij} = 0 \)
6. \( \nabla_i \nabla_j B^k + B^a R^a_{rij} = 2 \Omega_i \delta^j_k + 2 \Omega_j \delta^i_k = L_B \Gamma^k_{ij} \)
7. \( \nabla_i R^s_{jk} - \nabla_j R^s_{ki} = 0 \)
8. \( R^s_{jhd} \psi^h = 0 \)
9. \( \nabla_i \nabla_j D_k + D_a R^a_{ki} = 0 \)
10. \( C^h R^s_{ij} + C^s R^a_{ij} = 0 \)
11. \( \Omega_j = \frac{1}{4n} \nabla_i \nabla_j B^i, \Omega_j = \psi^j \)

where \( V = (v^k_b) = v^k E_k + v^k E_k^\perp, \Omega = \left( \Omega_j dx^i + \Omega_j dy^j \right) \).

**Proof.** Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let \( V \) be an infinitesimal paraholomorphically projective transformation with the associated 1-form \( \Omega \) on \( T^*M \)

\[ (L_V \nabla) (X, Y) = \Omega(X) Y + \Omega(Y) X + \Omega(JX) Y + \Omega(JY) X \]

for any \( X, Y \in \mathfrak{S}_0^1(M) \).

From

\[ (L_V \nabla) (E_i, E_j) = \Omega(E_i) E_j + \Omega(E_j) E_i + \Omega(JE_i) J E_j + \Omega(JE_j) J E_i \]

we obtain

\[ (L_V \nabla) (E_i, E_j) = 2 \left( \Omega_i \delta^j_k + \Omega_j \delta^i_k \right) E_k \] \( (5) \)

also

\[ (L_V \nabla) (E_i, E_j) = [\partial_i (\partial_j v^k)] E_k + [\partial_j (\partial_i v^k)] E_k \] \( (6) \)

from (5) and (6) we obtain

\[ \partial_i (\partial_j v^k) = 0 \Rightarrow v^k = p^s A^k_s + B^k \] \( (7) \)

and

\[ \partial_i (\partial_j v^k) = 2 \left( \Omega_i \delta^j_k + \Omega_j \delta^i_k \right). \] \( (8) \)

Contracting \( k \) and \( j \) in (8), we have

\[ \Omega_i = \partial_i \psi, \] \( (9) \)
where $\psi = \frac{1}{2n+2} \partial_j v^j$. If we use the expression (9) in (8), expression (8) is rewritten as follows:

\[
\partial_t \left( \partial_j v^k \right) = 2(\partial_t \psi)\delta^j_k + 2(\partial_j \psi)\delta^j_k.
\]

Differentiating (10) partially, we have

\[
\partial^k \partial_t \partial_j v^k = 2\partial^k \partial_t \psi \delta^j_k + 2\partial^k \partial_j \psi \delta^j_k
\]

\[
= 2\partial^k \partial_t \psi \delta^j_k + 2\partial^k \partial_j \psi \delta^j_k \delta^j_i
\]

\[
= \partial^k \partial_t (4\psi \delta^j_k)
\]

from here we get

\[
\partial^k \partial_t (\partial_j v^k - 4\psi \delta^j_k) = 0.
\]

Written here as

\[
M^j_k = \partial^k \left( \partial_j v^k - 4\psi \delta^j_k \right)
\]

(11)

and

\[
C^j_k + p a M^a_k = \partial_j v^k - 4\psi \delta^j_k,
\]

(12)

where $C^j_k$ and $M^j_k$ are certain functions which depend only on the variables $(x^h)$. Also

\[
M^j_k + M^{ji}_k = \partial^i \partial_j v^k - 4\partial^i \partial_j \psi \delta^j_k + \partial^j \partial_i v^k - 4\partial_j \psi \delta^j_k.
\]

Using (10) in above equation

\[
M^{ij}_k = \frac{1}{2} (M^{ij}_k - M^{ji}_k) = 2 \left( (\partial_j \psi) \delta^i_k - (\partial_i \psi) \delta^j_k \right).
\]

(13)

Contracting $k$ and $j$ in (12), we have

\[
C^k_k + p a M^{ak}_k = (2 - 2n)\psi.
\]

From which

\[
\psi = \frac{1}{2 - 2n} C^k_k + p a \frac{1}{2 - 2n} M^{ak}_k
\]

and we get

\[
\psi = \varphi + p a \Psi^a,
\]

(14)

where $\varphi = \frac{1}{2 - 2n} C^k_k$ and $\Psi^a = \frac{1}{2 - 2n} M^{ak}_k$, from which we have

\[
\Omega^i = \partial_t \psi = \Psi^i.
\]

(15)

If used (13) and (14) in (12) we get

\[
\partial_j v^k = C^j_k + 4\varphi \delta^j_k + 2p a \Psi^a \delta^j_k + 2p_k \Psi^j
\]

and

\[
v^k = D_k + p a C^a_k + 4\varphi p_k + 2p a p_k \Psi^a,
\]

(16)

where $D_k$ are certain functions which depend only on $(x^h)$. The coordinat transformation rule implies that $D = (D_k) \in S_1^0(M)$.

Next, from

\[
(L_{v^k}) (X, Y) = \Omega(X) Y + \Omega(Y) X + \Omega(JX) JY + \Omega(JY) JX,
\]

we have

\[
(L_{v^k}) (E^t_i, E^j_j) = 0
\]

or
\[(L_v^R \nabla) (E_i, E_j) = 0\]

from which, we get

\[0 = [\nabla_j A^{ki}] E_k + [A^i a p_s R^s_{kaj} + v^a R^i_{jak} + \nabla_j C^i_k + 2 p_k (\nabla_j \Psi^i) + 2 p_a \delta^i_k \nabla_j \Psi^a + 4(\partial_i \varphi) \delta^i_k] E_k.\]

Therefore,

\[\nabla_j A^{ki} = 0\] (17)

and

\[A^i a p_s R^s_{kaj} + v^a R^i_{jak} + \nabla_j C^i_k + 2 p_s (\nabla_j \Psi^i) + 2 p_a \delta^i_k \nabla_j \Psi^a + 4(\partial_i \varphi) \delta^i_k = 0.\] (18)

Contracting \(k\) and \(i\) in (18), we have

\[\begin{cases}
\nabla_j C^i_k = 0,
\n\nabla_j \varphi = 0
\end{cases}\] (19)

and

\[\nabla_j \Psi^s = \frac{1}{2(n+1)} A^i a R^s_{aij}.\] (20)

Lastly, from

\[(L_v^R \nabla) (E_i, E_j) = (2 \Omega_i \delta_j + 2 \Omega_j \delta_i) E_k\]

we obtain

\[(2 \Omega_i \delta_j + 2 \Omega_j \delta_i) E_k = [\nabla_i \nabla_j v^k + v^a R^k_{aij} + A^h k p_s R^s_{hji}] E_k + [p_s \left( (\nabla_i \Psi^s) R^s_{khj} + (\nabla_j \Psi^s) R^s_{khi} - (E_{R}^s v^k) R^s_{hji} \right) + v^a p_s (\nabla_i R^s_{fak} - \nabla_v R^s_{fji}) + (v^a R^s_{fki} + \nabla_i \nabla_j v^k)] E_k\]

from which, using (7) and (16), we obtain

\[\nabla_i \nabla_j R^k_{aij} + A^a R^k_{aij} + A^k R^h_{aij} = 0,\] (21)

\[\nabla_i \nabla_j B^k + B^a R^k_{aij} = 2 \Omega_i \delta_j + 2 \Omega_j \delta_i = L_B \Gamma^k,\] (22)

\[\nabla_i R^s_{fak} - \nabla_v R^s_{fji} = 0,\] (23)

\[\nabla_i \nabla_j \Psi^s + R^s_{fhi} \Psi^h = 0,\] (24)

\[\nabla_i \nabla_j D_k + D_a R^a_{kji} = 0,\] (25)

\[\nabla_i \nabla_j \Psi^s = \nabla_i \nabla_j R^s_{kai} + (\nabla_i B^a) R^s_{kai} + C^h R^s_{fhi} + C^a R^a_{kji} + \nabla_i \nabla_j C^s_k = 0.\] (26)

From (26), we get

\[K_{ij} = (\nabla_i B^a) R^s_{kai} + (\nabla_j B^a) R^s_{kaj} + C^h R^s_{fhi} + C^a R^a_{kji} + \nabla_i \nabla_j C^s_k = 0,\]

\[K_{ij} = (\nabla_i B^a) R^s_{kai} + (\nabla_j B^a) R^s_{kaj} + C^h R^s_{fhi} + C^a R^a_{kji} + \nabla_i \nabla_j C^s_k = 0.\]

Contracting \(j\) and \(k\) in (22), we obtain

\[\Omega_i = \frac{1}{4n} \nabla_i \nabla_j B^j.\] (28)

This completes the proof.

**Theorem 4** Let \((M, g)\) be a Riemannian manifold and \(T^*M\) be its cotangent bundle with the Riemannian
extension and adapted almost paracomplex structure. If $T^*M$ admits a non-affine infinitesimal paraholomorphically projective transformation, than $M$ and $T^*M$ are locally flat.

**Proof.** Let $V$ be non-affine infinitesimal paraholomorphically projective transformation on $T^*M$, using (3) in the expression of theorem 3, we have $\nabla_i\|\Psi\|^2 = \nabla_j\|\partial\psi\|^2 = 0$. Hence, $\|\Psi\|$ and $\|\partial\psi\|$ are constant on $M$. Suppose that $M$ is non-locally flat, then $\Psi = \partial\psi = 0$ by virtue of (9) and (3) in the expression of theorem 3, that is, $V$ is an infinitesimal affine transformation. This is a contradiction. Therefore, $M$ is locally flat. In this case $T^*M$ is locally flat.

**Corollary 5** Let $(M, g)$ be a Riemannian manifold and $T^*M$ be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vertical vector field $V$ is an infinitesimal paraholomorphically projective transformation with associated 1-form $\Omega$ on $T^*M$ if and only if there exist $D = (D^i_k) \in \mathfrak{X}^0_1(M)$ and $C = (C^i_k) \in \mathfrak{X}^1_1(M)$ satisfying

1. $\left( \begin{array}{c} \nabla^i_k \\ \nabla^j \\ \end{array} \right) = \left( \begin{array}{c} D^i_k + p_a C^a_k + 4q_k + 2p_k \Psi^a \\ 0 \\ \end{array} \right)$

2. $\nabla_i C^i_k = 0$

3. $\nabla_i \varphi = 0,$ $\nabla_i \psi = 0,$ $\nabla_i \Psi = 0$

4. $\nabla_i \nabla_j D^i = D^i_a R^a_{jkl} = 0$

5. $C^a_k R^a_{jkl} + C^i_k R^i_{jkl} = 0$

6. $\Psi^h R^h_{jkl} = 0$

7. $\Psi^a R^a_{jkl} + \Psi^b R^b_{jkl} = 0$

8. $\Omega_j = 0,$ $\Omega_j = \Psi^j$

where $V = \left( \begin{array}{c} 0 \\ \nabla^i_k \\ \nabla^j \\ \end{array} \right) = \psi^i E^i_k,$ $\Omega = \left( \Omega_j dx^j + \Omega_j \delta y^j \right)$.

**CONCLUSION**

In this article, we use the Levi-Civita connection of the Riemannian extension and we give definition and formulas almost paracomplex structure $J$. Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension $(^R\nabla)$ and adapted almost paracomplex structure $J$.

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