Dirac Equation in a 5-dimensional Kaluza-Klein Theory

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Abstract. Dirac equation is discussed in 5-dimensional space time having topology $M^4 \times T^1$, where M^4 and T^1 both are curved. It is shown that 4-dimensional fermion can be obtained from 5-dimensional fermion, as a result of compactification of extra dimension. It is found that the realistic 4-dimensional fermions are possible in higher modes earlier than those in lower modes during the course of expansion of 4-dimensional universe. 4-dimensional Dirac equation, obtained from 5-dimensional Dirac equation after compactification, is solved for an arbitrary modes for super-heavy as well as light (realistic) fermions. Time-dependence of polarization vector and magnetization density, as a result of Gordon-decomposition of the current vector for 4-dimensional spin-$\frac{1}{2}$ field (with arbitrary mode), is exhibited.

Keywords: Fermions, Dirac equation, Kaluza-Klein theory, energy- momentum tensor.

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1 Introduction

In the context of unification of gravity with gauge interactions, Kaluza-Klein type theories [1-4] are good candidates. In these theories, the spacetime is supposed to have the topology $M^4 \times T^{n-4}$ (n is the total number of dimension), where M^4 denotes the usual para-compact four-dim. spacetime (flat or curved) and T^{n-4} is the extra (n-4)-dim.compact manifold.The observed universe is 4-dimensional , hence it is supposed that the extra-manifold due to gauge interactions might be compact and very small in size so that these are not observed today. Physically, it is very much appealing to think that T^{n-4} manifold is curved due to its compact nature[5]. The action for higher-dim. gravity is $s = \int d^n x \sqrt{-g} R$ where $g = det\ g_{\mu\nu}$ ($g_{\mu\nu}$ is the metric tensor for the n-dimensional space time) and R is the Ricci curvature scalar for $M^4 \times T^{n-4}$. n-dimensional Einstein's equations derived from the above action yields that T^{n-4} can be curved (1) when M^4 is also curved or (2) action contains some lagrangian for matter field also so that energy-momentum tensor is non-vanishing[6]. This discussion shows that if M^4 is curved, the

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extra-dim. manifold will definitely be curved. Motivated by this idea, the line-element for 5-dim. space time is taken as

$$ds^2 = dt^2 - a^2(t)[dx_1^2 + dx_2^2 + dx_3^2] - A^2(t)dy^2 \tag{1.1}$$

Using the metric tensor from (1.1) in the Einstein's field equations[7] derived from the above action, it is interesting to see that

$$aA = 1 \tag{1.2}$$

Thus (1.1) deserves a model in which one spatial dimension shrinks with time while the other three expand. Earlier, Chodos and Detweiler[8] have considered this type of spacetime.

In 4-dim. spacetime, Dirac equation has been solved and discussed by many authors[9]. The aim of present paper is to discuss and solve Dirac equation for spin-$\frac{1}{2}$ field Ψ_5 in 5-dim. spacetime (1.1) for different time-function a(t).

The degrees of freedom for a spin-$\frac{1}{2}$ field Ψ in n dimensions[10] is given by 2^α where $\alpha = \frac{n}{2}$ (if n is even) and $\alpha = \frac{(n-1)}{2}$ (if n is odd). The dimension of the space-time, here, is 5, so degrees of freedom for Ψ is 4. The flat space Dirac matrices in 5-dim. will be $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_5$ where $\tilde{\gamma}_5 = \tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3$ (other $\gamma-$ matrices are the usual standard matrices for 4-dim.)[11]. Now for Weyl's transformation a matrix $\bar{\gamma} = \tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_5 = 1$ can be defined. So operation of chiral projection operators $\frac{1}{2}(1 \pm \bar{\gamma})$ on the Dirac equation

$$i\tilde{\gamma}^\mu \partial_\mu \Psi_5 + m_5 \Psi_5 = 0 \tag{1.3}$$

shows that chiral fermions are not possible in 5-dim.[10]. Hence m_5, the mass of Ψ_5 is not zero. This is true for fermions in curved spacetime also which obey the Dirac equation

$$i\gamma^\mu D_\mu \Psi_5 + m_5 \Psi_5 = 0 \tag{1.4}$$

where γ^μ are curved space Dirac-matrices and D_μ denotes covariant derivatives in curved spacetime (1.1).

The paper is planned as follows: Section 2 contains 5-dim. Dirac Equation in the space-time (1.1) and a brief discussion on the effective mass of 4-dim. fermion produced by 5-dim. fermion is given. Section 3 contains some explicit examples of solutions for 5-dim.

Dirac Equation. In the last section, Gordon-decomposition[12] of 4-dim. Ψ_4, obtained after compactification of 5.dim spacetime, has been discussed.
$\hbar = c = 1$ is used as fundamental unit where $\hbar$ and c carry their usual meaning. Here indices $a, \mu \cdots = 0, 1, 2, 3, 5$.

2. Dirac Equation for Ψ_5

The vierbein h^μ_a on the manifold $M^4 \times T^1 (T^1$ is circle) is defined as

$$h^\mu_a h^\nu_b g_{\mu\nu} = \eta_{ab} \tag{2.1}$$

where (μ, ν) are curved space indices, (a,b) are flat space indices , $g_{\mu\nu}$ is the curved space metric tensor and η_{ab} is the Minkowskian metric. So, in the space time (1.1)

$$h^0_0 = 1, h^1_1 = h^2_2 = h^3_3 = a^{-1}(t), h^5_5 = A^{-1}(t) = a(t) \tag{2.2}$$

The operator D_μ in (1.4) is defined as [13]

$$D_\mu = \partial_\mu - \Gamma_\mu \tag{2.3}$$

where Γ_μ is the spin coefficient or Ricci rotation coefficient given as

$$\Gamma_\mu = \frac{1}{4}\left(\partial_\mu h^\rho_a + \begin{Bmatrix} \rho \\ \sigma\mu \end{Bmatrix} h^\sigma_a\right) h^\nu_b g_{\rho\nu} \bar{\gamma}^a \bar{\gamma}^b \tag{2.4}$$

where $\left\{ {\rho \atop \sigma\mu} \right\}$ is the affine-connection.
$\tilde{\gamma}^a$ are flat space Dirac matrices satisfying the anti-commutation rule

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}. \tag{2.5}$$

Curved space Dirac matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.6}$$

γ^μ is related to $\tilde{\gamma}^a$ through vierbein as

$$\gamma^\mu = h^\mu_a \tilde{\gamma}^a. \tag{2.7}$$

So, Dirac equation in (1.1) is written as

$$i\left[\tilde{\gamma}^0(\partial_o + \frac{9}{4}\frac{\dot{a}}{a}) + a^{-1}(\bar{\gamma}^1\partial_1 + \bar{\gamma}^2\partial_2 + \bar{\gamma}^3\partial_3) + a\bar{\gamma}^5\partial_5\right]\Psi_5 - m_5\Psi_5 = 0, (\dot{a} = \partial_0 a) \tag{2.8}$$

The internal space is compactified, so

$$\Psi_5(x,y) = \sum_{n=-\infty}^{\infty} \phi_n(y)\Psi_4^{(n)}(x) \tag{2.9}$$

where $\Psi_4^{(n)}(x)$ are 4-dim. spinor fields in nth mode and $\phi_n(y)$ are harmonic functions which obey the equation[4]

$$i\tilde{\gamma}^5\partial_5\phi_n(y)\ \psi_4^{(n)} = \frac{n}{R}\phi_n(y)\ \psi_4^{(n)} \tag{2.10}$$

R is the compactification scale. ϕ_n is normalised as

$$\phi_n\phi_m = \delta_{nm}.$$

Connecting (2.8) and (2.10)

$$\left[i\{\tilde{\gamma}^0(\partial_0 + \frac{9\dot{a}}{4a}) + a^{-1}(\tilde{\gamma}^1\partial_1 + \tilde{\gamma}^2\partial_2 + \bar{\gamma}^3\partial_3)\} - (m_5 - \frac{an}{R})\right]\Psi_4^n = 0 \tag{2.11}$$

(2.11) shows that effective mass for Ψ_4^n is

$$M = m_5 - \frac{an}{R} = m_5 - anM_c \tag{2.12}$$

where $R^{-1} = M_c$ is the compactification mass.

The Extra dimension y which is assumed to be a circle of radius R, has the range

$$0 \leq y \leq 2\pi R \tag{2.13}$$

The distance around the extra dimension is given by [13]

$$\delta_5 = \int_o^{2\pi R} dy\sqrt{-g_{55}} = \frac{2\pi R}{a(t)} \tag{2.14}$$

which is time-dependent and decreases as $a(t)$ increases.
At $t = t_c$ (compactification time), it is assumed that

$$\frac{2\pi R}{a(t_c)} \sim L_P \ \ (\text{PlankLength}) \tag{2.15}$$

For realistic fermions, $M \simeq 0$. So, as a particular time $t = t_P$

$$m_5 \simeq \frac{na(t_p)}{R} \tag{2.16}$$

Connecting (2.15) and (2.16)

$$m_5 \simeq \frac{2\pi n a(t_p) M_P}{a(t_c)} \tag{2.17}$$

where M_P is the Planck mass and is equal to $(L_P)^{-1}$

Hence from (2.17)

$$n \simeq \frac{m_5 a(t_c)}{2\pi a(t_p) M_P} \leq \frac{m_5}{2\pi M_P} \tag{2.18}$$

as $a(t_p) \gtrsim a(t_c)$ in the expanding 4-dim. universe. So (2.18) puts a constraint on modes n. Also (2.18) states that if one gets realistic 4-dim. fermion from 5-dim fermion as a result of compactification of the extra dimension, number of physically allowed modes depends on m_5. For example, $n = 0, 1$ if $m_5 = 2\pi M_p$; $n = 0, 1, 2$ if $m_5 = 4\pi M_p$; $n = 0, 1, 2, ...(r-1), r$ if $m_5 = 2\pi r M_p$ where r is positive integer. From (2.16)

$$a(t_p) \backsimeq \frac{m_5 R}{n} \tag{2.19}$$

So, one gets

$$a(t_p) \backsimeq \frac{2\pi r M_p R}{n} \tag{2.20}$$

where $n = 0, 1, 2, ...(r-1), r$. Now it is interesting to see that if $a(t_{p1}) = a_1$ when $n = r$ and $a(t_{p2}) = a_2$ when $n = r - 1, a_2 > a_1$. It means that realistic fermions may be obtained in higher modes earlier than those in lower modes during the course of expansion of the obsevable 4-dim. universe, as $t_{p1} < t_{p2}$.

3. Solution of Dirac Equation

Substituting Ψ defined as

$$\psi_4^{(n)} = a^{-\frac{9}{4}} \Psi \tag{3.1}$$

(2.11) is re-written as

$$\left[(\tilde{\gamma}^0 \partial_\tau + \tilde{\gamma}^1 \partial_1 + \tilde{\gamma}^2 \partial_2 + \tilde{\gamma}^3 \partial_3) + ia(m_5 - anM_c) \right] \Psi = 0 \tag{3.2}$$

with

$$\tau = \int^t \frac{dt'}{a(t')} \tag{3.3}$$

Writing

$$\Psi = \sum_k \Psi_k = (2\pi)^{-\frac{3}{2}} exp(ikx) \begin{bmatrix} f_I(\vec{k}, \tau) \\ f_{II}(\vec{k}, \tau) \end{bmatrix} \tag{3.4}$$

One gets two coupled equations

$$[\partial_\tau + ia(m_5 - anM_c)]f_I + i(k.\sigma)f_{II} = 0 \tag{3.5a}$$

$$[\partial_\tau - ia(m_5 - anM_c)]f_{II} + i(k.\sigma)f_I = 0 \tag{3.5b}$$

where $\sigma_1, \sigma_2, \sigma_3$ are pauli matrices.

From (3.5a)

$$f_{II} = ik^{-2}(k.\sigma)[\partial_\tau + ia(m_5 - anM_c)]f_I \tag{3.6}$$

Connecting (3.5b)and (3.6)

$$f_I'' + [k^2 + a^2(m_5 - anM_c)^2 + i\partial_\tau\{a(m_5 - anM_c)\}]f_I = 0 \tag{3.7}$$

where prime (/) denotes differentiation with respect to τ.

The norm for ψ_5 is defined at τ=constant hypersurface as

$$(\psi_{5ks}, \psi_{5k's'}) = \int_\tau \sqrt{|g_5|}d^3x dy\, \bar{\psi}_{5ks}\gamma^0\psi_{5k's'} \tag{3.8}$$

where g_5 is the determinant of 5-dim. metric tensor.

Connecting (2.9) and (3.8)

$$(\psi_{5ks}, \psi_{5k's'}) = \int_\tau \sqrt{|g_5|}d^3x dy \sum_n \bar{\psi}^n_{4ks}(x)\bar{\phi}_n(y)\gamma^0 \sum_{n'} \phi_{n'}(y)\psi^{n'}_{4k's'}(x) \tag{3.9}$$

which yields, on integration over extra dimension y having the range $0 \leq y \leq 2\pi R$

$$(\psi_{5ks}, \psi_{5k's'}) = 2\pi R \int_\tau \sqrt{|g_5|}d^3x \sum_n \bar{\psi}^n_{4ks}\gamma^0\psi^n_{4k's'} \tag{3.10}$$

The normalization constants are determined using the prescription that in the flat space limit

$$(\psi_{5ks}, \psi_{5k's'}) \longrightarrow \delta_{ss'}\delta^3(k - k') \tag{3.11}$$

or,

$$2\pi R \int_\tau \sqrt{|g_5|}d^3x \sum_n \bar{\psi}^n_{4ks}\gamma^0\psi^n_{4k's'} \longrightarrow \delta_{ss'}\delta^3(k - k') \tag{3.12}$$

Now, some explicit examples of solutions of Dirac equation for ψ^n_4 utilising the above mentioned procedure of normalization are given as under for different $a(t)$.

$3(A)\quad a(t) \simeq t^{\frac{1}{2}}$

From (3.3)

$$\tau = 2t^{\frac{1}{2}} \ \ and \ \ a(t) = \frac{1}{2}\tau, \tag{3.13}$$

so (3.7) is written as

$$f_I'' + \left[k^2 + \frac{\tau^2}{4}\left(m_5 - \frac{nM_c\tau}{2}\right)^2 + \frac{i}{2}\partial_\tau\{\tau(m_5 - \frac{nM_c\tau}{2})\}\right] f_I = 0 \tag{3.14}$$

(3.14) can be approximated in two different ways:

<u>**Case I:**</u> When $m_5 >> \frac{1}{2}nM_c\tau$ (3.14) is approximated as

$$f_I'' + \left[k^2 + \frac{im_5}{2} - \frac{inM_c\tau}{4} + \frac{m_5^2\tau^2}{4}\right] f_I = 0 \tag{3.15}$$

which has exact solution

$$f_I = exp\left[\pm\frac{nM_c}{4m_5}\tau \mp \frac{im_5}{4}\tau^2\right] \ [N_1 \ {}_1F_1(a, \frac{1}{2}, X) + \\ N_2\frac{e^{i(6r+1)\frac{\pi}{4}}}{\sqrt{2m_5}}(\frac{nM_c}{2m_5} - im_5\tau) \ {}_1F_1(\frac{1}{2} + a, \frac{3}{2}, X)] \tag{3.16}$$

where

$$a = \pm\frac{i}{m_5}\left(\frac{n^2M_c^2}{32m_5^4} + \frac{k^2}{2m_5^2}\right),$$

$$X = \pm\frac{i}{2m_5}\left(\frac{nM_c}{2m_5} - im_5\tau\right)^2,$$

${}_1F_1(a, c, X)$ is confluent hypergeometric function and $r = 0, 1, 2, \ldots$.

Connecting (3.6)and(3.16)

$$f_{II} = \frac{i(k.\sigma)}{k^2}exp\left[\pm\frac{nM_c}{4m_5}\tau \mp \frac{im_5}{4}\tau^2\right] \times \\ [N_3\{(\pm\frac{nM_c}{4m_5} \mp \frac{im_5}{2}\tau + \frac{im_5}{2}\tau - \frac{in\tau^2M_c}{4}) \ {}_1F_1(a, \frac{1}{2}, X) + 2aX' \ {}_1F_1(1 + a, \frac{3}{2}, X)\} + \\ N_4e^{i(6r+1)\frac{\pi}{4}}\{(\frac{nM_c}{2m_5} - im_5\tau)(\pm\frac{nM_c}{4m_5} \mp \frac{im_5}{2}\tau + \frac{im_5}{2}\tau - \frac{in\tau^2M_c}{4}) \ {}_1F_1(\frac{1}{2} + a, \frac{3}{2}, X) - \\ i\sqrt{\frac{m_5}{2}} {}_1F_1(\frac{1}{2} + a, \frac{3}{2}, X) + \frac{(1 + 2a)}{3}X'\frac{(\frac{nM_c}{2m_5} - im_5\tau)}{\sqrt{2m_5}} {}_1F_1(\frac{3}{2} + a, \frac{5}{2}, X)\}] \tag{3.17}$$

Corresponding to f_I and f_{II}, ψ^n_{k4Is} and ψ^n_{k4IIs} are written as

$$\psi^n_{k4Is} = (2\pi)^{-\frac{3}{2}}(\frac{\tau}{2})^{-\frac{9}{4}}exp[ikx \pm \frac{nM_c}{4m_5}\tau \mp \frac{im_5\tau^2}{4}] \times [N_1 \ u_s \ {}_1F_1(a, \frac{1}{2}, X) + \\ N_2\hat{u}_s e^{i(6r+1)\pi/4}\frac{(\frac{nM_c}{2m_5} - im_5\tau)}{\sqrt{2m_5}} {}_1F_1(\frac{1}{2} + a, \frac{3}{2}, X)] \tag{3.18}$$

$$\psi^n_{k4IIs} = (2\pi)^{-\frac{3}{2}}(\frac{\tau}{2})^{-\frac{9}{4}}\frac{i(k.\sigma)}{k^2}exp[ik.x \pm \frac{nM_c}{4m_5}\tau \mp \frac{im_5\tau^2}{4}]\times$$
$$[N_3 u_s\{(\pm\frac{nM_c}{4m_5} \mp \frac{im_5}{2}\tau + \frac{im_5}{2}\tau - \frac{in\tau^2 M_c}{4})\ _1F_1(a,\frac{1}{2},X) + 2aX'\ _1F_1(1+a,\frac{3}{2},X)\}+$$
$$N_4\hat{u}_s e^{i(6r+1)\pi/4}\{\frac{(\frac{nM_c}{2m_5} - im_5\tau)}{\sqrt{2m_5}}(\pm\frac{nM_c}{4m_5} \mp \frac{im_5}{2}\tau + \frac{im_5}{2}\tau - \frac{in\tau^2 M_c}{4})_1F_1(\frac{1}{2}+a,\frac{3}{2},X)-$$
$$i\sqrt{\frac{m_5}{2}}\,_1F_1(\frac{1}{2}+a,\frac{3}{2},X) + \frac{(1+2a)}{3}X'\frac{(\frac{nM_c}{2m_5} - im_5\tau)}{\sqrt{2m_5}}\,_1F_1(\frac{3}{2}+a,\frac{5}{2},X)\}] \qquad (3.19)$$

where u_s and $\hat{u}_s$ are column matrices (with spin quantum number s=±1) given as

$$\hat{u}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \hat{u}_{-1} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$(3.20)$$

$$u_1 = \begin{pmatrix}0\\0\\-k_3\\-k_1-ik_2\end{pmatrix} \quad \text{and} \quad u_{-1} = \begin{pmatrix}0\\0\\-k_1+ik_2\\k_3\end{pmatrix}$$

Normalization constants N_1,N_2,N_3 and N_4 are determined through (3.12) as

$$N_1 = \frac{\sqrt{M_c}exp(\mp nM_c/2m_5)}{2k\sqrt{\pi}\mid\ _1F_1(a,\frac{1}{2},\overline{X})\mid}$$

$$N_2 = \frac{\sqrt{M_c}exp(\mp nM_c/2m_5)}{2\sqrt{\pi}\sqrt{\frac{n^2M_c^2+16m_5^2}{8m_5^3}}\mid\ _1F_1(a,\frac{1}{2}+a,\frac{3}{2},\overline{X})\mid}$$

$$N_3 = \frac{\sqrt{M_c}exp\,(\mp nM_c/2m_5)}{2\sqrt{\pi}\ s_1}$$

$$N_4 = \frac{\sqrt{M_c}exp\,(\mp nM_c/2m_5)}{2k\sqrt{\pi}\ s_2}$$

where

$$\overline{X} = \pm\left(\frac{nM_c}{2m_5} - 2im_5\right)$$

$$S_1 = \mid(\pm\frac{nM_c}{4m_5} \mp im_5 + im_5 - inM_c)_1F_1(a,\frac{1}{2},\overline{X}) \pm 2a(\frac{nM_c}{2m_5} - im_5)\ _1F_1(1+a,\frac{3}{2},\overline{X})\mid$$

and

$$S_2 = \mid \frac{(\frac{nM_c}{2m_5} - 2im_5)}{\sqrt{2m_5}}(\pm \frac{nM_c}{4m_5} \mp im_5 + im_5 - inM_c)_1F_1(\frac{1}{2}+a, \frac{3}{2}, \overline{X}) -$$

$$i\sqrt{\frac{m_5}{2}}\,{}_1F_1(\frac{1}{2}+a, \frac{3}{2}, \overline{X}) + \frac{(1+2a)}{3}\frac{(\frac{nM_c}{2m_5} - 2im_5)}{\sqrt{2m_5}}\,{}_1F_1(\frac{3}{2}+a, \frac{5}{2}, \overline{X}) \mid$$

Case II When $m_5 \gtrsim \frac{n\tau}{2}M_c$, a new variable τ' is defined as $\tau' = m_5 - \frac{nM_c}{2}\tau$. Now (3.14) is written as

$$\frac{d^2 f_I}{d\tau'^2} + \frac{4}{n^2M_c^2}[k^2 + \frac{1}{n^2M_c^2}(m_5 - \tau')^2\tau'^2 - \frac{i}{2}\frac{d}{d\tau'}\{\tau'(m_5 - \tau')\}]f_I = 0$$

Since $m_5 >> \tau'$, so one gets

$$\frac{d^2 f_I}{d\tau'^2} + [\frac{4}{n^2M_c^2}(k^2 - \frac{im_5}{2}) + \frac{4m_5^2}{n^4M_c^4}\tau'^2]f_I = 0 \tag{3.21}$$

having the exact solution

$$f_I = exp(-\frac{z}{2})[\tilde{N}_1\,{}_1F_1(k, \frac{1}{2}, z) + \tilde{N}_2 z^{1/2}\,{}_1F_1(k+\frac{1}{2}, \frac{3}{2}, z)] \tag{3.22}$$

where

$$z = \frac{im_5}{2}(\tau - \frac{2m_5}{nM_c})^2$$

and

$$4k = 1 + \frac{2i}{m_5}(k^2 - \frac{im_5}{2})$$

Connecting (3.6) and (3.22)

$$f_{II} = \frac{i(k.\sigma)}{k^2}exp(-\frac{z}{2})[\tilde{N}_3\{(\frac{1}{2}z' - \frac{inM_c\tau^2}{4} + \frac{im_5\tau}{2})\,{}_1F_1(k, \frac{1}{2}, z) +$$
$$2kz'\,{}_1F_1(k+1, \frac{3}{2}, z)\} + \tilde{N}_4\{z^{1/2}(\frac{1}{2}z' - \frac{inM_c}{4}\tau^2 + \frac{im_5\tau}{2} + \frac{z'}{2z})\,{}_1F_1(k+\frac{1}{2}, \frac{3}{2}, z)$$
$$+ \frac{(2k+1)}{3}z'z^{1/2}\,{}_1F_1(k+\frac{3}{2}, \frac{5}{2}, z)\}] \tag{3.23}$$

Corresponding to f_I and f_{II}, ψ^n_{k4Is} and ψ^n_{k4IIs} as are written as

$$\psi^n_{k4Is} = (2\pi)^{-3/2}exp[ik.x](\frac{\tau}{2})^{-9/4}[\tilde{N}_1 u_s\,{}_1F_1(k, \frac{1}{2}, z) + \tilde{N}_2\hat{u}_s z^{1/2}\,{}_1F_1(k+\frac{1}{2}, \frac{3}{2}, z)] \tag{3.24}$$

$$\psi^n_{k4IIs} = (2\pi)^{-3/2}exp[ik.x](\frac{\tau}{2})^{-9/4}e^{-z/2}\frac{i(k.\sigma)}{k^2}\times$$
$$[\tilde{N}_3 u_s\{(\frac{1}{2}z' - \frac{inM_c\tau^2}{4} + \frac{inM_5\tau}{2})\,{}_1F_1(k, \frac{1}{2}, z) + 2kz'\,{}_1F_1(k+1, \frac{3}{2}, z)\}$$
$$+ \tilde{N}_4\hat{u}_s\{z^{1/2}(\frac{1}{2}z' - \frac{inM_c\tau^2}{4} + \frac{inM_5\tau}{2} + \frac{z'z^{-1}}{2})\,{}_1F_1(k+\frac{1}{2}, \frac{3}{2}, z) +$$
$$\frac{(2k+1)}{3}z'z^{\frac{1}{2}}\,{}_1F_1(k+\frac{3}{2}, \frac{5}{2}, z)\}] \tag{3.25}$$

On normalizing these solutions

$$\tilde{N}_1 = \sqrt{M_c}\ [2k\sqrt{\pi}\ \mid {}_1F_1(k, \frac{1}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \mid]^{-1}$$

$$\tilde{N}_2 = n\sqrt{M_c}\ [2\sqrt{\pi}(nM_c - m_5)\sqrt{2m_5} \mid\ {}_1F_1(k + \frac{1}{2}, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \mid]^{-1}$$

$$\tilde{N}_3 = \frac{n\sqrt{M_c}}{2\sqrt{\pi}}(\tilde{s_1})^{-1}$$

$$\tilde{N}_4 = \frac{kn\sqrt{M_c}}{2\sqrt{n}}(\tilde{s_2})^{-1}$$

where

$$\tilde{S}_1 = \mid \{im_5(nM_c - m_5) - in^2M_c^2 + inm_5M_c\}{}_1F_1(k, \frac{1}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2 \\ + 4ikm_5(nM_c - m_5){}_1F_1(k + 1, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \mid$$

and

$$(\tilde{S}_2)^2 = m_5(1 - \frac{m_5}{nM_c}) \mid \{\frac{im_5(nM_c - m_5)}{nM_c} - inM_c + im_5 + \frac{nM_c}{2(nM_c - m_5}\}\times \\ {}_1F_1(k + \frac{1}{2}, \frac{3}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) + \\ \frac{(2k+1)}{3}\frac{2im_5(nM_c - m_5)}{nM_c}{}_1F_1(k + \frac{3}{2}, \frac{5}{2}, \frac{2im_5}{n^2M_c^2}(nM_c - m_5)^2) \mid$$

3B. $a(t) \simeq t$

From (3.3), in this case

$$\tau = lnt, \qquad a(t) = exp(\tau) \tag{3.26}$$

Connecting (3.7) and (3.26)

$$f_I'' + [k^2 + e^{2\tau}(m_5 - nM_ce^{\tau})^2 + ie^{\tau}(m_5 - nM_ce^{\tau}) - inM_ce^{2\tau}]f_I = 0 \tag{3.27}$$

Case 1 When $m_5 >> nM_c\ e^{\tau}$, (3.27) is approximated as

$$f_I'' + [k^2 + im_5e^{\tau} + (m_5^2 - inM_c)e^{2\tau}]f_I = 0 \tag{3.28}$$

which yields the exact solutions

$$f_I(\tau) = exp[\mp ik\tau + (inM_c - m_5^2)e^{\tau}] \times [c_1\ {}_1F_1(\frac{2ll' + im_5}{2l'}, 2l, -\frac{e^{\tau}}{2l'}) + \\ c_2(-\frac{e^{\tau}}{2l'})^{1-2l}\ {}_1F_1(\frac{2l' - 2ll' + im_5}{2l'}, 2 - 2l, -\frac{e^{\tau}}{2l'})] \tag{3.29}$$

where

$$l = \frac{1}{2}[1 \pm i2k], \quad l' = inM_c - m_5^2$$

Connecting (3.6) and (3.29)

$$f_{II}(\tau) = \frac{i(k.\sigma)}{k^2} exp[\mp ik\tau + (inM_c - m_5^2)e^\tau][c_3 X(\tau) + c_4 Y(\tau)] \tag{3.30}$$

where

$$X(\tau) = \{\mp ik + inM_c - m_5^2 - ie^\tau(nM_c e^\tau - m_5)\} \ {}_1F_1(\frac{2ll' + im_5}{2l'}, 2l, -\frac{e^\tau}{2l'}) - \frac{(2ll' + im_5)}{8ll'^2} \ {}_1F_1(\frac{2ll' + im_5 + 2l'}{2l'}, 1 + 2l, -\frac{e^\tau}{2l'})$$

and

$$Y(\tau) = (-\frac{e^\tau}{2l'})^{1-2l}[\{1 - 2l \mp ik + inM_c - m_5^2 - ie^\tau(nM_c e^\tau - m_5)\}\times \ {}_1F_1(\frac{2l' - 2ll' + im_5}{2l'}, 2 - 2l, -\frac{e^\tau}{2l'}) - \frac{(2l' - 2ll' + im_5)}{8l'^2(1-l)} \ {}_1F_1(\frac{4l' - 2ll' + im_5}{2l'}, 3 - 2l, -\frac{e^\tau}{2l'})]$$

So,

$$\psi^n_{k4Is} = (2\pi)^{-3/2} exp[ik.x - \frac{9\tau}{4} \mp ik\tau + (inM_c - m_5^2)e^\tau]\times [c_1 u_s \ {}_1F_1(\frac{2ll' + im_5}{2l'}, 2l, -\frac{e^\tau}{2l'}) + c_2 \hat{u}_s (-\frac{e^\tau}{2l'})^{1-2l} \ {}_1F_1(\frac{2l' - 2ll' + im_5}{2l'}, 2 - 2l, -\frac{e^\tau}{2l'})] \tag{3.31}$$

and

$$\psi^n_{k4IIs} = \frac{i(k.\sigma)}{k^2}(2\pi)^{-3/2} exp[ik.x - \frac{9\tau}{4} \mp ik\tau + (inM_c - m_5^2)e^\tau]\times [c_3 u_s X(\tau) + c_4 \hat{u}_s Y(\tau)] \tag{3.32}$$

On normalising these solutions

$$c_1 = \frac{\sqrt{M_c} exp(m_5^2)}{2\sqrt{\pi} k \mid \ {}_1F_1(\frac{2ll' + im_5}{2l'}, 2l, -\frac{1}{2l'}) \mid}$$

$$c_2 = \frac{\sqrt{M_c}(2l')^{1-2l} exp(m_5^2)}{2\sqrt{\pi} \mid \ {}_1F_1(\frac{2l' - 2ll' + im_5}{2l'}, 2 - 2l, -\frac{1}{2l'}) \mid}$$

$$c_3 = \frac{\sqrt{M_c} exp(m_5^2)}{2\sqrt{\pi} \mid X(0) \mid}$$

$$c_4 = \frac{\sqrt{M_c} exp(m_5^2)}{2\sqrt{\pi} \mid Y(0) \mid}$$

where $X(\tau)$ and $Y(\tau)$ are defined as in (3.30)

Case II. When $m_5 \gtrsim nM_c\, e^\tau$, (3.27) is approximated a

$$f_I'' + [k^2 - inM_c e^{2\tau}] f_I = 0 \tag{3.33}$$

which integrates to

$$\begin{aligned} f_I = exp(\pm ik\tau + l'e^\tau)[\tilde{c}_1\, {}_1F_1(\frac{2ll' - inM_c}{2l'}, 2l, -2l'e^\tau) + \\ \tilde{c}_2(-2l')^{1-2l} e^{\tau(1-2l)}\, {}_1F_1(\frac{2l' - 2ll' - inM_c}{2l'}, 2 - 2l, -2l'e^\tau)] \end{aligned} \tag{3.34}$$

where $L = \frac{1}{2}[1 \pm 2ik]$ and $L' = \sqrt{nM_c} \times exp[(2r+1)\frac{\pi}{4}]$ with $r = 0, 1, 2, \ldots$ (3.6) and (3.34) yield

$$f_{II} = \frac{i(k.\sigma)}{k^2} exp(\pm ik\tau + L'e^\tau)[\tilde{c}_3 \tilde{X}(\tau) \qquad \tilde{c}_4 \tilde{Y}(\tau)] \tag{3.35}$$

where

$$\begin{aligned} \tilde{X}(\tau) = \{\pm ik\tau + L'e^\tau + ie^\tau(m_5 - nM_c e^\tau)\}{}_1F_1(\frac{2LL' - inM_c}{2L'}, 2L, -2L'e^\tau) \\ - \frac{(2LL' - inM_c)}{2L} e^\tau\, {}_1F_1(\frac{2L' + 2LL' - inM_c}{2L'}, 1 + 2L, -2L'e^\tau) \end{aligned}$$

and

$$\begin{aligned} \tilde{Y}(\tau) = \{\pm ik\tau + L'e^\tau + (-2L')^{1-2L}(2 - 2L)e^{\tau(1-2l)} + ie^\tau(m_5 - inM_c e^\tau)\} \times \\ {}_1F_1(\frac{2L' - 2LL' - inM_c}{2L'}, 2 - 2L, -2L'e^\tau) + \\ (-2L')^{1-2L} \frac{(2L' - 2LL' - inM_c)}{2(L-1)} e^{2(1-L)\tau} \times \\ {}_1F_1(\frac{4L' - 2LL' - inM_c}{2L'}, 3 - 2L, -2L'e^\tau) \end{aligned}$$

So,

$$\begin{aligned} \psi_{k4Is}^n = (2\pi)^{-3/2} exp[ikx - \frac{9\tau}{4} \pm ik\tau + L'e^\tau] \times \\ [\tilde{c}_1 u_s\, {}_1F_1(\frac{2LL' - inM_5}{2L'}, 2L, -2L'e^\tau) + \\ \tilde{c}_2 \hat{u}_s (-2L')^{1-2L} e^{\tau(1-2L)}\, {}_1F_1(\frac{2L' - 2LL' - inM_c}{2L'}, 2 - 2L, -2L'e^\tau)] \end{aligned} \tag{3.36}$$

and

$$\psi^n_{k4IIs} = \frac{i(k.\sigma)}{k^2}(2\pi)^{-3/2} exp[ik.x - \frac{9\tau}{4} \pm ik\tau + L'e^{\tau}]\times$$
$$[\tilde{c}_3 u_s \tilde{X}(\tau) + \tilde{c}_4 \hat{u}_s \tilde{Y}(\tau)] \quad (3.37)$$

where $\tilde{X}(\tau)$ and $\tilde{Y}(\tau)$ are defined in (3.35)

On normalizing the above solutions

$$\tilde{c}_1 = \frac{e^{-L'}\sqrt{M_c}}{k\sqrt{2\pi} \mid {}_1F_1(\frac{2LL'-inM_c}{2L'}, 2L, -2L') \mid}$$

$$\tilde{c}_2 = \frac{e^{-L'}\sqrt{M_c}(-2L')^{(L-1)}}{\sqrt{2\pi} \mid {}_1F_1(\frac{2L'-2LL'-inM_c}{2L'}, 2-2L, -2L') \mid}$$

$$\tilde{c}_3 = \frac{e^{-L'}\sqrt{M_c}}{\sqrt{2\pi} \mid \tilde{X}(0) \mid}$$

$$\tilde{c}_4 = \frac{ke^{-L'}\sqrt{M_c}}{\sqrt{2\pi} \mid \tilde{Y}(0) \mid}$$

<u>$3C. \quad a(t) = e^{xt}$</u>

From (3.3)

$$-x\tau = e^{-xt}, \qquad a(\tau) = -(x\tau)^{-1} \quad (3.38)$$

so, (3.7) reduces to

$$f''_I + [k^2 + \frac{1}{x^2\tau^2}(m_5 + \frac{nM_c}{x\tau})^2 - i\frac{d}{d\tau}\{\frac{1}{x\tau}(m_5 + \frac{nM_c}{x\tau})\}]f_I = 0 \quad (3.39)$$

<u>Case 1</u> When $m_5 >> \frac{nM_c}{x\tau}$ (3.39) is approximated to

$$\tau^2 f''_I + [k^2\tau^2 + \frac{m_5(m_5 + ix)}{x^2}]f_I = 0 \quad (3.40)$$

having exact solution

$$f_I = exp[\pm ik\tau + \frac{1}{2}(1 \pm i\sqrt{4a^2-1})ln\tau][D_1\ {}_1F_1(\frac{a}{2}, 2g, -2g'\tau)$$
$$+ D_2(-2g'\tau)^{\pm i\sqrt{4a^2-1}}\ {}_1F_1(1 + \frac{a}{2} - 2g, 2 - 2g, -2g'\tau)] \quad (3.41)$$

where

$$a = m_5(m_5 + ix)x^{-2} \quad , \quad g = \frac{1}{2}[1 \pm \sqrt{4a^2-1}]$$

and $g' = \mp ik$.

Connecting (3.6) and(3.41)

$$f_{II} = \frac{i(k.\sigma)}{k^2} exp[\pm ik\tau + \frac{1}{2}(1 \pm i\sqrt{4a^2-1})ln\tau][D_3 X_1(\tau) + D_4 Y_1(\tau)] \quad (3.42)$$

where

$$X_1(\tau) = \{\frac{1 \pm \sqrt{4a^2-1}}{2\tau} \pm ik - \frac{i}{x\tau}(\frac{nM_c}{x\tau} + m_5) - \frac{ag'}{2g}\}\ {}_1F_1(\frac{a}{2}, 2g, -2g'\tau)$$

and

$$Y_1(\tau) = (-2g'\tau)^{\pm i\sqrt{4a^2-1}}[\{\frac{1 \pm i\sqrt{4a^2-1}}{2\tau} \pm ik - \frac{i}{x\tau}(\frac{nM_c}{x\tau} + m_5) + \tau^{-1}\} \times$$
$${}_1F_1(1 + \frac{a}{2} - 2g, 2 - 2g, -2g'\tau) - \frac{(2 + a - 4g)}{2(1-g)} g'\ {}_1F_1(\frac{4 + a - 4g}{2}, 3 - 2g, -2g'\tau)]$$

So,

$$\psi^n_{k4Is} = (2\pi)^{-3/2} e^{i(k.x)}(-\frac{1}{x\tau})^{-9/4} exp[\pm ik\tau + \frac{1}{2}(1 \pm i\sqrt{4a^2-1})ln\tau] \times$$
$$[D_1\ u_s\ {}_1F_1(\frac{a}{2}, 2g, -2g'\tau) +$$
$$D_2 \hat{u}_s (-2g'\tau)^{\pm i\sqrt{4a^2-1}}\ {}_1F_1(\frac{2 + a - 4g}{2}, 2 - 2g, -2g'\tau)] \quad (3.43)$$

and

$$\psi^n_{k4IIs} = (2\pi)^{-3/2} e^{i(k.x)}(-\frac{1}{x\tau})^{-9/4} exp[\pm ik\tau + \frac{1}{2}(1 \pm i\sqrt{4a^2-1})ln\tau] \times$$
$$\frac{i(k.\sigma)}{k^2}[D_3 u_s X_1(\tau) + D_4 \hat{u}_s Y_1(\tau)] \quad (3.44)$$

Normalization of these solutions yields

$$D_1 = \sqrt{M_c x}\ e^{-i\pi/2}[2\sqrt{\pi}\ k\ |\ {}_1F_1(\frac{a}{2}, 2g, \frac{2g'}{x})\ |]^{-1}$$

$$D_2 = \sqrt{M_c x}\ e^{-i\pi/2}[2\sqrt{\pi}\ \ |\ {}_1F_1(\frac{2 + a - 4g}{2}, 2 - 2g, \frac{2g'}{x})\ |]^{-1}$$

$$D_3 = \sqrt{M_c x}\ e^{-i\pi/2}[2\sqrt{\pi}\ \ |\ X_1(-\frac{1}{x})\ |]^{-1}$$

and

$$D_4 = k\sqrt{M_c x}\ e^{-i\pi/2}[2\sqrt{\pi}\ \ |\ Y_1(-\frac{1}{x})\ |]^{-1}$$

<u>Case II</u> When $m_5 \gtrsim nM_c x\tau$, (3.39) approximates to

$$\tau^2 f''_I + [\alpha + \beta\tau + \gamma\tau^2] f_I = 0 \quad (3.45)$$

where $\alpha = m_5^2 x^{-2}$, $\beta = m_5^2(2-ix)(nM_c)^{-1}$
and $\gamma = k^2 + m_5^3(m_5 - ix)(nM_c)^{-2}$
(3.45)yields the solution

$$f_I = exp[\pm i\sqrt{\gamma}\tau + \frac{1}{2}(1 \pm i\sqrt{4\alpha - 1})ln\tau] \times [\tilde{D}_1 \, {}_1F_1(\frac{2jj'+\beta}{2j'}, 2j, -2j'\tau) + \tilde{D}_2(-2j'\tau)^{1-2j} \, {}_1F_1(\frac{2j' - 2jj' + \beta}{2j'}, 2-2j, -2j'\tau)] \quad (3.46)$$

where $j = \frac{1}{2}[1 \pm i\sqrt{4\alpha - 1}]$ and $j' = \pm i\sqrt{\gamma}$
Connecting (3.6) and (3.46)

$$f_{II} = \frac{i(k.\sigma)}{k^2} exp[\pm i\sqrt{\gamma}\tau + \frac{1}{2}(1 \pm i\sqrt{4\alpha - 1})ln\tau] \times [\tilde{D}_3 X_2(\tau) + \tilde{D}_4 Y_2(\tau)] \quad (3.47)$$

where

$$X_2(\tau) = \{\frac{1}{2\tau}(1 \mp i\sqrt{4\alpha - 1}) \mp i\sqrt{\gamma} - \frac{i}{x\tau}(\frac{nM_c}{x\tau} + m_5)\} \, {}_1F_1(\frac{2jj'+\beta}{2j'}, 2j, -2j'\tau) - \frac{(2jj'+\beta)}{2j} \, {}_1F_1(\frac{2jj'+\beta+2j'}{2j'}, 1+2j, -2j'\tau)$$

and

$$Y_2(\tau) = [\frac{1}{2\tau}(1 \mp i\sqrt{4\alpha - 1}) \mp i\sqrt{\gamma} - \frac{i}{x\tau}(\frac{nM_c}{x\tau} + m_5)](-2j'\tau)^{1-2j} \times {}_1F_1(\frac{2j' - 2jj' + \beta}{2j'}, 2-2j, -2j'\tau) - 2j(1-2j)(-2j\tau)^{-2j} \, {}_1F_1(\frac{2j' - 2jj' + \beta}{2j'}, 2-2j, -2j'\tau) - (-2j\tau)^{1-2j}\frac{(2j - 2jj' + \beta)}{2(1-j)} \, {}_1F_1(\frac{4j' - 2jj' + \beta}{2j'}, 3-2j, -2j'\tau)$$

So

$$\psi^n_{k4Is} = (2\pi)^{-3/2} e^{i(k.x)} exp[\pm i\sqrt{\gamma}\tau + \frac{1}{2}(1 \pm i\sqrt{4\alpha - 1})ln\tau] \times [\tilde{D}_1 \, u_s \, {}_1F_1(\frac{2jj'+\beta}{2j'}, 2j, -2j'\tau) + \tilde{D}_2 \hat{u}_s(-2j\tau)^{1-2j} \, {}_1F_1(\frac{2j' - 2jj' + \beta}{2j'}, 2-2j, -2j'\tau)] \quad (3.48)$$

and

$$\psi^n_{k4IIs} = (2\pi)^{-3/2} e^{ik.x} \frac{i(k.\sigma)}{k^2} exp[\pm i\sqrt{\gamma}\tau + \frac{1}{2}(1 \pm i\sqrt{4\alpha - 1})ln\tau] \times [\tilde{D}_3 \, u_s X_2(\tau) + \tilde{D}_4 \, u_s Y_2(\tau)] \quad (3.49)$$

On normalization of these solutions

$$\tilde{D}_1 = \sqrt{M_c x}\; e^{-i\pi/2}[2k\sqrt{\pi} \mid {}_1F_1(\frac{2jj'+\beta}{2j'}, 2j, \frac{2j'}{x}) \mid]^{-1}$$

$$\tilde{D}_2 = \sqrt{M_c x}\; e^{-i\pi/2}[2\sqrt{\pi} \mid (\frac{2j}{x})^{1-2j}\; {}_1F_1(\frac{2j' - 2jj' + \beta}{2j'}, 2-2j, \frac{2j'}{x}) \mid]^{-1}$$

$$\tilde{D}_3 = \sqrt{M_c x}\; e^{-i\pi/2}[2\sqrt{\pi} \mid X_2(-\frac{1}{x}) \mid]^{-1}$$

and

$$\tilde{D}_4 = \sqrt{M_c x}\; e^{-i\pi/2}k[2\sqrt{\pi} \mid Y_2(-\frac{1}{x}) \mid]^{-1}$$

4. Current for ψ_4^n

The current is defined as

$$J_4^{\hat{\mu}n} = \bar{\psi}_4^n \gamma^{\hat{\mu}} \psi_4^n, \quad (\hat{\mu} = 0,1,2,3) \tag{4.1}$$

which is divergence - free as $J_4^{\hat{\mu}n}; \hat{\mu} = 0$. For a massive field

$$J_4^{\hat{\mu}n} = \bar{\psi}_4^n \gamma^{\hat{\mu}} \psi_4^n = \frac{1}{2M} \bar{\psi}_4^n (i\partial_{\hat{\lambda}} \gamma^{\hat{\lambda}} \gamma^{\hat{\mu}} - i\gamma^{\hat{\mu}} \gamma^{\hat{\lambda}} \partial_{\hat{\lambda}} - i[\gamma^{\hat{\lambda}} \Gamma_{\hat{\lambda}}, \gamma^{\hat{\mu}}]) \psi_4^n \tag{4.2}$$

where $M = m_5 - a(t)nM_c$

which can be re-expressed as

$$\begin{aligned} J_4^{n\hat{\mu}} = & \frac{1}{2M}(\bar{\psi}_4^n \sigma^{\hat{\lambda}\hat{\mu}} \psi_4^n), \hat{\lambda} - \frac{i}{4M} g^{\hat{\mu}\hat{\lambda}} \bar{\psi}_4^n \overleftrightarrow{\partial_{\hat{\lambda}}} \psi_4^n \\ & - \frac{i}{4M} \bar{\psi}_4^n ([\gamma^{\hat{\lambda}}, _{\hat{\lambda}}, \gamma^{\hat{\mu}}] + [\gamma^{\hat{\lambda}}, \gamma^{\hat{\mu}}, _{\hat{\lambda}}]) \psi_4^n \\ & - \frac{i}{2M} \bar{\psi}_4^n [\gamma^{\hat{\lambda}} \Gamma_{\hat{\lambda}}, \gamma^{\hat{\mu}}] \psi_4^n \end{aligned}$$

where

$$\bar{\psi}_4^n \overleftrightarrow{\partial_{\hat{\lambda}}} \psi_4^n = \bar{\psi}_4^n \partial_{\hat{\lambda}} \psi_4^n - \psi_4^n \partial_{\hat{\lambda}} \bar{\psi}_4^n, \quad M = m_5 - \frac{a(t)n}{k_c}$$

(here $\hat{\mu}, \hat{\nu}, \hat{\lambda}, \ldots$ run from 0 to 3)

In the M^4 spacetime

$$\gamma^{\hat{\lambda}}, _{\hat{\lambda}} = 0, \quad [\gamma^{\hat{\lambda}}, \gamma^i, _{\hat{\lambda}}] = [\tilde{\gamma}^0, \tilde{\gamma}^i](-\dot{a}\bar{a}^2) \qquad (i = 1,2,3)$$

$$[\tilde{\gamma}^0, \gamma^0, _0] = 0, \quad \sigma^{0i} = \frac{i}{2a}[\tilde{\gamma}^0, \tilde{\gamma}^i], \quad \sigma^{ij} = \frac{i}{2a^2}[\tilde{\gamma}^i, \tilde{\gamma}^j]$$

$$\Gamma_0 = 0, \quad \Gamma_1 = \dot{a}\tilde{\gamma}^1\tilde{\gamma}^0, \quad \Gamma_2 = \dot{a}\tilde{\gamma}^2\tilde{\gamma}^0 \;\; and \;\; \Gamma_3 = \dot{a}\tilde{\gamma}^3\tilde{\gamma}^0$$

So

$$J_4^{no} = \frac{1}{2M}(\bar{\psi}_4^n \sigma^{io} \psi_4^n)_{,i} - \frac{i}{4M} \bar{\psi}_4^n \overleftrightarrow{\partial_0} \psi_4^n \tag{4.3}$$

and

$$\begin{aligned} J_4^{ni} = & \frac{1}{2M}\partial_0(\bar{\psi}_4^n \sigma^{oi} \psi_4^n) + \frac{1}{2M}\partial_j(\bar{\psi}_4^n \sigma^{ji} \psi_4^n) + \\ & \frac{7i}{2M}(\frac{\dot{a}}{a^2})\bar{\psi}_4^n \tilde{\gamma}^0 \tilde{\gamma}^i \psi_4^n + \frac{i}{4Ma^2}\bar{\psi}_4^n \overleftrightarrow{\partial_i} \psi_4^n \end{aligned} \tag{4.4}$$

In terms of polarization density and magnetization density J_4^{no} and J_4^{ni} is written as

$$J_4^{no} = \vec{\nabla}.\vec{p}_4^n + \rho_{4(convective)}^n$$

and

$$J_4^{ni} = \partial_t \vec{p}_4^n + \nabla \times \vec{M}_4^n + \vec{j}_{4(convective)}^n + 7(\frac{\dot{a}}{a})\vec{p}_4^n \tag{4.5}$$

When $m_5 >> \frac{an}{kc}$, $M \simeq m_5$, so $P_4^{in} = \frac{i}{2m_5 a}\bar{\psi}_4^n \tilde{\gamma}^i \tilde{\gamma}^o \psi_4^n$

and

$$M_4^{in} = \epsilon^{ijk}(\frac{i}{4m_5 a^2})\bar{\psi}_4^n [\tilde{\gamma}_j, \tilde{\gamma}_k] \psi_4^n$$

$$\rho_{4(convective)}^n = -\frac{i}{4m_5}\bar{\psi}_4^n \overleftrightarrow{\partial_0} \psi_4^n$$

and

$$J_{4(convective)}^{in} = -\frac{i}{4m_5 a^4}\bar{\psi}_4^n \overleftrightarrow{\partial^i} \psi_4^n \tag{4.6}$$

But when $m_5 \gtrsim \frac{an}{R_c}$, one has

$$P_4^{in} = \frac{i}{2(m_5 - \frac{an}{R_c})a}\bar{\psi}_4^n \tilde{\gamma}^i \tilde{\gamma}^o \psi_4^n$$

$$M_4^{in} = \epsilon^{ijk}[\frac{i}{4(m_5 - \frac{an}{R_c})a^2}]\bar{\psi}_4^n [\tilde{\gamma}_j, \tilde{\gamma}_k] \psi_4^n$$

$$\rho_{4(convective)}^n = -[\frac{i}{4(m_5 - \frac{an}{R_c})}]\bar{\psi}_4^n \overleftrightarrow{\partial_0} \psi_4^n$$

and

$$J_{4(convective)}^{in} = \frac{ian}{2a(m_5 R - an)}\bar{\psi}_4^n \tilde{\gamma}^i \tilde{\gamma}^o \psi_4^n - \frac{iR}{4(m_5 R - an)a^4}\bar{\psi}_4^n \overleftrightarrow{\partial^i} \psi_4^n \tag{4.7}$$

Thus, it is found that polarization vector, magnetization density (which is a pseudo-vector),$\rho_{(convective)}$ and $J_{(convective)}$[9] , depend on time . It is interesting to note that when $m_5 \gtrsim \frac{an}{R_c}$ (which yields realistic fermions) $J_{(convective)}$ contains an extra term $\frac{ian}{2a(m_5 R - an)}\bar{\psi}_4^n \tilde{\gamma}^i \tilde{\gamma}^o \psi_4^n$.

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