CONGRUENCES OF CURVES IN A RIEMANNIAN SPACE

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Congruences of curves in a Euclidean space of three dimensions were studied by WEATHERBURN. In this paper the author has obtained the generalised expressions for the focal surfaces, limit surfaces and some other expressions for a set of $m-n$ congruences of curves, which are such that through each point of a subspace V_n of n dimensions immersed in a Riemannian manifold V_m of m dimensions $(m>n)$, one curve of each congruence passes.

1. Focal Hypersurfaces. Foci.

Let a V_n of coordinates x^i $(i=1, 2, \ldots, n)$ (1) be immersed in a Riemannian space V_m of coordinates y^α $(\alpha=1, 2, \ldots, m)$. Suppose the metrics of V_n and V_m are positive definite and are given by $g_{ij}\, dx^i\, dx^j$ and $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$ respectively. Then we have [3]

$$g_{ij} = a_{\alpha\beta}\, y^\alpha{}_{;i}\, y^\beta{}_{;j}, \tag{1.1}$$

where a semi colon (;) followed by a Latin index denotes tensor derivatives with respect to x's.

Let us consider a set of $m-n$ congruences of curves in V_m which are such that one curve of each congruence passes through each point of the subspace V_n. Let $s_{\tau|}$ $(\tau=n+1, \ldots, m)$ be the length of a curve of a congruence-$\lambda_{\tau|}$ measured from the point P at which the curve intersects V_n to another point Q on the curve. The subspace V_n will henceforth be known as the subspace of reference.

Now, we consider two adjacent curves of the congruence-$\lambda_{\tau|}$, when the curves approach each other at one or more points, the infinitesimal normal vector, normal to both the curves, is of the second or higher order. Thus the infinitesimal distance from a point

$$y^\alpha{}_{\tau|}\,(x^1, \ldots, x^n, s_{\tau|})$$

on the former curve to another point

$$y^\alpha{}_{\tau|}\,(x^1+dx^1,\ x^2+dx^2,\ \ldots,\ x^n+dx^n,\ s_{\tau|}+ds_{\tau|})$$

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(1) Throughout this paper we adopt the convention that Latin letters take the values $1, 2, \ldots, n$; early letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta$ etc.) take values $1, 2, \ldots, m$; and later letters ($\mu, \nu, \sigma, \varrho$ etc.) the values $n+1, \ldots m$.

on the latter is of of the second or higher order. Hence, neglecting quantities of the second or higher order, we get

$$y^{\alpha}{}_{\tau|}(x^1, x^2, \ldots, x^n, s_{\tau|}) = y^{\alpha}{}_{\tau|}(x^1 + dx^1, \ldots, x^n + dx^n, s_{\tau|} + ds_{\tau|})$$

so that

$$\frac{\partial y^{\alpha}{}_{\tau|}}{\partial x^i} dx^i + \frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}} ds_{\tau|} = 0. \tag{1.2}$$

For a fixed τ, (1.2) consists of m equations in $n+1$ unknown quantities dx^i and $ds_{\tau|}$.

We shall now consider the following particular cases:

Case 1: When $m = n + 1$, *i. e.* the immersed space is a hypersurface, then the set of $m - n$ congruences will reduce to a single congruence, we shall henceforth call it the congruence-λ. In this case through each point of the hypersurface only one curve of the congruence will pass. As τ can have only one value, we may drop it, and the equation (1.2) can be written as

$$\frac{\partial y^{\alpha}}{\partial x^i} dx^i + \frac{\partial y^{\alpha}}{\partial s} ds = 0, \tag{1.3}$$

which is satisfied when the vectors with components $\partial y^{\alpha}/\partial x^1, \ldots, \partial y^{\alpha}/\partial x^n$, and $\partial y^{\alpha}/\partial s$ all lie in the same geodesic surface, *i. e.*

$$\frac{\partial y^{\alpha}}{\partial x^1} \frac{\partial y^{\alpha}}{\partial x^2} \cdots \frac{\partial y^{\alpha}}{\partial x^n} \frac{\partial y^{\alpha}}{\partial s} = 0. \tag{1.4}$$

The expression on the left side of this equation stands for the determinants

$$\varepsilon_{\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}} \frac{\partial y^{\alpha_1}}{\partial x^1} \cdots \frac{\partial y^{\alpha_r}}{\partial x^r} \cdots \frac{\partial y^{\alpha_{n+1}}}{\partial s}, \tag{1.5}$$

where $\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}$ take all the values from 1 to $n+1$ and [1]

$$\varepsilon_{\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}} = 0,$$

if at least two of the indices $\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}$ are equal;

$$\varepsilon_{\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}} = 1,$$

if these indices are all different and constitute a permutation of even order with respect to the fundamental permutation $1, 2, \ldots, n+1$; and

$$\varepsilon_{\alpha_1 \ldots \alpha_r \ldots \alpha_{n+1}} = -1,$$

if the indices are all different and constitute a permutation of odd order.

We observe that (1.4) is the equation of a hypersurface, let us call it the *focal* *hypersurface* by analogy with the focal surface of a rectilinear congruence in a Euclidean space of three dimensions.

The points in which a focal hypersurface is met by a curve of the congruence-λ, will be known as the *foci* of the congruence. The number of foci on any curve depends upon the degree of g^{α} in s. Supposing that g^{α} is a polynomial of degree p in s, then the equations (1.4) is of degree $p(n+1)-1$ in s and there are $p(n+1)-1$ foci on each curve. When

(I) $p=1$, the number of foci on each curve is n. Hence :

If g^{α} is linear in s, the number of foci on each curve of the congruence is the same as the number of dimensions of the hypersurface of referenc.

(II) *a* When the congruence of curves is in a Euclidean space of three dimensions, we have

$$n=2,$$

therefore there are $3p-1$ foci on each curve. This result has been obtained by WEATHERBURN [2].

(II) *b* In case the rectilinear congruence is in a Euclidean space of three dimensions, we have

$$p=1, \quad \text{and} \quad n=2$$

and there are two foci on each line of the congruence.

Case 2: If $m=n+1$, *i. e.* the immersed space is a subspace, then the equations (1.2) will not be consistent, therefore no foci will exist. Hence we have the theorem :

A set of $m-n$ congruences of curves in a Riemannian V_m, which are sach that through each point of a subspace V_n of V_m one curve of each congruence passes, have no foci, provided the subspace is not a hypersurface.

These results are in complete analogy with the results of a Euclidean space of three dimensions. A rectilinear congruence is referred to a surface of reference, the coordinates of a point on which can be expressed as functions of two parameters; whereas the coordinates of a point in space are expressed as functions of three parameters. We have in this case definite focal points. If we take simply the generators of a ruled surface referred to a directrix, the coordinates of which are expressed as functions of one parameter only, we do not have foci on the generators. That is, if the generators are referred to a variety whose dimensions are one less than the enveloping space, we have no foci; but if they are referred to a variety whose dimensions are less by a number greater than unity, we do not have foci.

The cases 1 and 2 are reminiscent of the results of Euclidean space of three dimensions.

2. Limit points. Limit Hypersurface.

Let

$$y^{\alpha}{}_{\tau|} \quad \text{and} \quad y^{\alpha}{}_{\tau|}+y^{\alpha}{}_{\tau|;i}\,dx^{i}+\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\,ds_{\tau|}$$

be respectively the contravariant components of the position vectors of two points M and M' on two adjacent curves of the congruence and let the contravariant components of the unit tangent vectors to the curves at M and M' be $\partial y^{\alpha}{}_{\tau|}/\partial s_{\tau|}$ and

$$\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}+\left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\right)_{;i}dx^{i}+\frac{\partial^{2} y^{\alpha}{}_{\tau|}}{\partial s^{2}{}_{\tau|}}\,ds_{\tau|}$$

respectively. If MM' be the infinitesimal vector perpendicular to $\partial y^{\alpha}{}_{\tau|}/\partial s_{\tau|}$ and

$$\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}+\left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\right)_{;i}dx^{i}+\frac{\partial^{2} y^{\alpha}{}_{\tau|}}{\partial s^{2}{}_{\tau|}}\,\partial s_{\tau|}$$

we have

$$a_{\alpha\beta}\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\cdot\left(y^{\beta}{}_{\tau|;i}\,dx^{i}+\frac{\partial y^{\beta}{}_{\tau|}}{\partial s_{\tau|}}\,ds_{\tau|}\right)=0, \tag{2.1}$$

and

$$\bar{a}_{\alpha\beta}\left\{\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}+\left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\right)_{;i}dx^{i}+\frac{\partial^{2} y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}{}^{2}}\,ds_{\tau|}\right\}\cdot\left\{y^{\beta}{}_{\tau|;j}\,dx_{j}+\frac{\partial y^{\beta}{}_{\tau|}}{\partial s_{\tau|}}\,ds_{\tau|}\right\}=0, \tag{2.2}$$

where $a_{\alpha\beta}$ and $\bar{a}_{\alpha\beta}$ are the covariant components of the fundamental tensor of the enveloping space at M and M' respectively.

From (2.1) we obtain

$$ds_{\tau|}=-\,a_{\alpha\beta}\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\,y^{\beta}{}_{\tau|;i}\,dx^{i}$$

or

$$ds_{\tau|}=-\,\bar{p}_{\tau|i}\,dx^{i} \tag{2.3}$$

where

$$\bar{p}_{\tau|i}=a_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial s_{\tau|}}\,y^{\beta}{}_{\tau|;i}\,. \tag{2.4}$$

Neglecting the terms of the second order of smallness, from (2.2) and (2.3) we obtain

$$\bar{a}_{\alpha\beta}\left\{\left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\right)_{;i}y^{\beta}{}_{\tau|;j}-\left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}}\right)_{;i}\frac{\partial y^{\beta}{}_{\tau|}}{\partial s_{\tau|}}\,\bar{p}_{\tau|j}-\frac{\partial^{2} y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}{}^{2}}\,\bar{p}_{\tau|i}\,y^{\beta}{}_{\tau|j}\right\}dx^{i}\,dx^{j}=0 \tag{2.5}$$

or

$$\bar{A}_{\tau|ij}\, dx^i\, dx^j = 0, \tag{2.6}$$

where

$$\bar{A}_{\tau|ij} = \bar{a}_{\alpha\beta} \left\{ \left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}} \right)_{;i} z^{\beta}{}_{\tau|;j} - \left(\frac{\partial y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}} \right)_{;i} \frac{\partial z^{\beta}{}_{\tau|}}{\partial s_{\tau|}} \bar{F}_{\tau|j} - \frac{\partial^2 y^{\alpha}{}_{\tau|}}{\partial s_{\tau|}^2} y^{\beta}{}_{\tau|;j} \bar{F}_{\tau|i} \right\}. \tag{2.7}$$

The curve of the congruence-$\lambda_{\tau|}$ intersects the subspace given by (2.6) in a number of points which we call the *central points* of the congruence-$\lambda_{\tau|}$ by analogy with the central points of a rectilinear congruence in a Euclidean space of three dimensions. The values of $s_{\tau|}$ given by the equation (2.6) determine the distances of the central points from the subspace of reference. These distances depend on the ratio $dx^i : dx^j$. The maximum and minimum values of these distances are given by differentiating (2.6) with respect to dx^j; so that

$$\bar{A}_{\tau|ij}\, dx^i = 0, \tag{2.8}$$

which is satisfied, when

$$\text{Det.}\ |\, \bar{A}_{\tau|ij} \,| = 0. \tag{2.9}$$

The values of $s_{\tau|}$ obtained from (2.9) determine the points which correspond to limit points of a rectilinear congruence in a Euclidean space of three dimonsions. For want of more suitable name, we call them *limits* or *limit points*. The equation (2.9) represents the locus of the limit points; we call it the *limit surface*.

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ÖZET

Üç boyutlu bir Öklid uzayındaki eğri kongrüansları WEATHERBURN tarafından incelenmiştir. Bu yazıda ise $m-n$ eğri kongrüansından ibaret bir eğri sisteminin odak yüzeyi, limit yüzeyi ve daha başka bazı özel kavramları için ifadeler elde edilmiştir. Bütün bu sonuçlara varmak için m boyutlu V_m RIEMANN uzayına daldırılmış olan her n boyutlu V_n $(m>n)$ ait uzayının her noktasından her kongrüansın bir ve bir tek eğrisinin geçtiği farzedilmektedir.