PROPAGATION OF SMALL DISTURBANCES IN A VISCOELASTIC FLUID CONTAINED BETWEEN TWO COAXIAL CYLINDERS

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The propagation of small disturbances in a viscoelastic fluid contained between two coaxial cylinders due to the slow angular motion of its base is discussed in two cases: (i) the angular velocity of the disc is given by $\omega(t) = \omega_0\delta(t)$ (Dirac-delta function); (ii) $\omega(t) = \omega_0 \sin(nt)$. The solutions as λ (the relaxation time parameter) tends to zero are shown to correspond to those for an ordinary viscous fluid. It has been observed that unlike the ordinary viscous fluid, the velocity of propagation of the disturbances in case (i) is finite and the disturbance does not reach all points of the fluid instantaneously.

1. Introduction. Various authors such as ROBERTS [6], LUNDQUIST [5], BHATNAGAR and KUMAR [1], KUMAR [3,4] have discussed the propagation of small disturbances in viscous incompressible fluids and in inviscid and electrically conducting fluids in the presence of a magnetic field.

The author elsewhere [7,8] has discussed the propagation of disturbances in au idealized viscoelastic fluid in two cases:

(*i*) The fluid occupies the space ($z > 0$). The disturbance is produced by the slow angular motion of a disc $x^2 + y^2 = a^2$, $z = 0$.

(*ii*) The fluid is contained in an infinite circular cylinder. The disturbance is produced by the slow angular motion of its base.

In this paper, when the relaxation phenomenon of the fluid is considered, we study the propagation of disturbances in the viscoelastic fluid contained in the annular space $z \geq 0$, $b \leq r \leq a$. The disturbance is produced by giving the base of the annulus a slow angular motion about the z-axis at time $t = 0$ which ia represented for time $t > 0$ by $\omega(t)$, where

(*i*) $\omega(t) = \omega_0\, \delta(t)$ ($\delta(t)$ is the DIRAC-delta function),

(*ii*) $\omega(t) = \omega_0 \sin nt$.

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It is interesting to note that as λ (the relaxation time) tends to zero, the solutions give the solution of the corresponding problem for an ordinary viscous fluid as obtained earlier [3].

2. Formulation of the problem. The stress-strain rate relation for an idealized incompressible visco-elastic fluid when only the relaxation phenomenon is taken into consideration is of the form

(1) $$s_{ik} + \lambda\left[\frac{\partial}{\partial t} s_{ik} + v_j s_{ik,j} - v_{i,j} s_{jk} - v_{k,j} s_{ij}\right] = 2\mu\, e_{ik},$$

where λ is the relaxation time constant and μ is the coefficient of viscosity. The equations of motion in the absence of extraneous forces are

(2) $$\varrho\left[\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right] = -p_{,i} + s_{ij,j},$$

ϱ being the density of the fluid, and the equation of continuity is

(3) $$v_{i,i} = 0.$$

We take cylindrical polar coordinates (r, ϑ, z). We assume $v_r = v_z = 0$ and $\frac{\partial}{\partial \vartheta} = 0$ (due to symmetry about z-axis). Then the equation of continuity is satisfied identically. We thus get v (the component of velocity in the ϑ-direction as $v(r, z, t)$.

Equation (1) when transformed to cylindrical polar coordinates, gives under these assumptions, the following equations

(4) $$s_{rr} + \lambda \frac{\partial}{\partial t}(s_{rr}) = 0,$$

(5) $$s_{rz} + \lambda \frac{\partial}{\partial t}(s_{rz}) = 0,$$

(6) $$s_{zz} + \lambda \frac{\partial}{\partial t}(s_{zz}) = 0,$$

(7) $$s_{\vartheta z} + \lambda \frac{\partial}{\partial t}(s_{\vartheta z}) = \mu \frac{\partial v}{\partial z},$$

(8) $$s_{r\vartheta} + \lambda \frac{\partial}{\partial t}(s_{r\vartheta}) = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

and

(9) $$s_{\vartheta\vartheta} + \lambda \frac{\partial}{\partial t}(s_{\vartheta\vartheta}) - 2\lambda\, s_{r\vartheta}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = 0.$$

Equations (4), (5) and (6) give

(10) $$s_{rr} = 0, \quad s_{rz} = 0, \quad s_{zz} = 0.$$

Then from (2) we get

(11) $$\frac{\partial p}{\partial z} = 0,$$

(12) $$\frac{\partial p}{\partial r} + \frac{s_{\vartheta\vartheta}}{r} = \frac{\varrho v^2}{r},$$

and

(13) $$\varrho \frac{\partial v}{\partial t} = \frac{\partial}{\partial r}(s_{r\vartheta}) + \frac{\partial}{\partial z}(s_{\vartheta z}) + \frac{2}{r} s_{r\vartheta}$$

From equations (7), (8) and (13) we get the following equation for determining v

(14) $$\lambda \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} = \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right],$$

where $\nu = (\mu/\varrho)$ is the kinematic coefficient of viscosity.

3. Solution of the problem. The fluid contained in the annular space $z \geq 0$, $a \leq r \leq a$ is at rest. At the instant $t = 0$ the disturbance is initiated by giving the base of the anuulus a slow angular motion about the z-axis which at later times is defined by $\omega(t)$.

Therefore we solve equation (14) under the following initial and boundary conditions.

a) Initial conditions

(15) $$v = 0, \quad t \leq 0,$$

(16) $$\frac{\partial v}{\partial t} = 0, \quad t \leq 0,$$

b) Boundary conditions

(17) $$v = r\,\omega(t), \quad b \leq r \leq a, \quad z = 0,$$

(18) $$v = 0, \qquad r = a, \qquad z \geq 0,$$

(19) $$v = 0, \qquad r = b, \qquad z \geq 0,$$

(20) $$v = 0 \qquad \text{when} \qquad z \to \infty,$$

Case (i). Let

$$\omega(t) = \omega_0\, \delta(t),$$

where $\delta(t)$ is the Dirac delta function.

Multiplying equation (14) by e^{-pt} and integrating between the limits $t=0$ to $t=\infty$, we get

$$(21) \qquad (\lambda p^2 + p)\,\bar{v} = \nu \left[\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r^2} + \frac{\partial^2 \bar{v}}{\partial z^2} \right]$$

where

$$(22) \qquad \bar{v} = \int_0^\infty e^{-pt}\, v\, dt$$

is the LAPLACE transform of v.

Let

$$(23) \qquad \bar{v}_H = \int_b^a r\, K(r, a, \xi_i)\, \bar{v}\, dr$$

be the finite HANKEL transform of $\bar{v}$, where

$$(24) \qquad K(r, a, \xi_i) = J_1(\xi_i r)\, G_1(\xi_i a) - J_1(\xi_i a)\, G_1(\xi_i r)$$

and ξ_i is a positive root of the equation

$$(25) \qquad J_1(\xi b)\, G_1(\xi a) - G_1(\xi b)\, J_1(\xi a) = 0.$$

Multiplying equation (21) by $r\,K(r, a, \xi_i)$ and integrating between the limits b to a we get

$$(26) \qquad \frac{d^2 \bar{v}_H}{dz^2} = \left[\frac{\lambda}{\nu}\, k \left(p + \frac{1}{\lambda} \right) + \xi_i^2 \right] \bar{v}_H\,.$$

We solve equation (26) under the boundary conditions

$$(27) \qquad \bar{v}_H = 0 \quad \text{when} \quad z = \infty$$

and

$$(28) \qquad \bar{v}_H = \omega_0 \int_b^a r^2\, K(r, a, \xi_i)\, dr$$

$$= \frac{\omega_0}{\xi_i^2} \left[\frac{b\, J_1(a\,\xi_i)}{J_1(b\,\xi_i)} - a \right] \quad \text{when} \quad z = 0.$$

The solution of equation (26) is

$$(29) \qquad \bar{v}_H = \frac{\omega_0}{\xi_i^2} \left[\frac{b\, J_1(a\,\xi_i)}{J_1(b\,\xi_i)} - a \right] e \;\; \text{exp.} \left\{ - \left[\frac{\lambda}{\nu}\, p \left(p + \frac{1}{\lambda} \right) + \xi_i^2 \right]^{1/2} z \right\}.$$

Taking the HANKEL inversion of (29) and then using the inversion theorem for LAPLACE transform, we get

(30) $$v = 2\omega_0 \sum_i \left[\frac{b\, J_1(a\,\xi_i)}{J_1(b\,\xi_i)} - a\right] \frac{J_i^2(b\,\xi_i)\, K(r, a, \xi_i)}{J_i^2(a\,\xi_i) - J_i^2(b\,\xi_i)}$$

$$\left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e \quad \exp.\left\{- pt - z\left[\frac{\lambda}{\nu}\, p\left(p+\frac{1}{\lambda}\right)+\xi_i^2\right]^{1/2}\right\}\right. \qquad j \text{ being } \sqrt{-1}.$$

We split

(31) $$\sum_{i=1}^{\infty} \quad \text{into} \quad \sum_{i=1}^{k} + \sum_{i=k+1}^{\infty}$$

where k is such that

(32) $$\xi_k < \frac{1}{2\sqrt{\lambda\nu}} < \xi_{k+1}.$$

Now on evaluating the p-integral [2] in (30) we get when

$$t > \left(\frac{\lambda z^2}{\nu}\right)^{1/2}$$

(33) $$v = \frac{\omega_0 \left(\frac{z^2\lambda}{\nu}\right)^{1/2} e^{-t/2\lambda}}{\lambda\left(t^2 - \frac{z^2\lambda}{\nu}\right)^{1/2}} \left[\sum_{i=1}^{\infty} \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{\{J_i^2(a\,\xi_i) - J_i^2(b\,\xi_i)\}\,(1-4\,\lambda\nu\xi_i^2)^{-1/2}}\, I_1 \left\{\frac{(1-4\,\lambda\,\nu\,\xi_i^2)^{1/2}\left(t^2 - \frac{\lambda z^2}{\nu}\right)^{1/2}}{2\lambda}\right\} - \sum_{r=k+1}^{\infty} \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{\{J_i^2(a\,\xi_i) - J_i^2(b\,\xi_i)\}\,(4\lambda\,\nu\,\xi_i^2 - 1)^{-1/2}}\, J_1 \left\{\frac{(4\lambda\nu\xi_i^2-1)^{1/2}\left(t^2 - \frac{\lambda z^2}{\nu}\right)^{1/2}}{2\lambda}\right\}\right]$$

and

(34) $$v = 0$$

when

$$t < \left(\frac{\lambda z^2}{\nu}\right)^{1/2}.$$

Now we proceed to take the limit as $\lambda \to 0$. As $\lambda \to 0$ the second summation in (33) vanishes and we get

$$v_{vis} = \lim_{\lambda \to 0} v = \frac{\omega_0 z e^{-(z^2/4\nu t)}}{(\pi\nu)^{1/2} t^{3/2}} \sum_{i=1}^{\infty} \frac{\{b J_1(a\xi_i) - a J_1(b\xi_i)\} J_1(b\xi_i) K(r, a, \xi_i) e^{-\nu t \xi_i^2}}{[J_1^2(a\xi_i) - J_1^2(b\xi_i)]}. \tag{35}$$

This limiting value of v denoted by v_{vis} corresponds to the solution of the problem of an ordinary viscous incompressible fluid as already obtained [8].

It is interesting to note that (33) is valid when

$$t > \left(\frac{z^2\lambda}{\nu}\right)^{1/2}$$

We may have a physical picture of this in the following manner. Keeping $z = z_0$ we see that so far as

$$t < \left(\frac{z_0^2\lambda}{\nu}\right)^{1/2}$$

the disturbance has not reached the region $z \geq z_0$. Thus we conclude that the disturbance is not propagated instantaneously in the whole region occupied by the fluid as in the case of an ordinary viscous fluid but it has a finite velocity of propagation and reaches any particular point $z = z_0$ only after $t = t_0$ where

$$t_0 = \left(\frac{z_0^2\lambda}{\nu}\right)^{1/2}.$$

Case (ii). We take $\omega(t) = \omega_0 \sin nt$.

Applying the same techniques as in case (i) we get the same equation (26) for determining $\bar{v}_H$.

We solve equation (26) under the boundary conditions

$$\bar{v}_H = 0, \quad \text{when} \quad z = \infty \tag{36}$$

and

$$\bar{v}_H = \frac{\omega_0 n}{p^2 + n^2} \frac{b J_1(a\xi_i) - a J_1(b\xi_i)}{\xi_i^2 J_1(b\xi_i)} \quad \text{when} \quad z = 0 \tag{37}$$

and get

$$\bar{v}_H = \frac{n\omega_0}{p^2 + n^2} \cdot \frac{b J_1(a\xi_i) - a J_1(b\xi_i)}{\xi_i^2 J_1(b\xi_i)} e \ \exp. \ -\left\{\frac{\lambda}{\nu} p \left(p + \frac{1}{\lambda}\right) + \xi_i^2\right\}^{1/2} z. \tag{38}$$

Taking the HANKEL inversion of (38) and then finding the LAPLACE inversion we get

(39) $$v = 2\omega_0 \sum_i \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{[J_1^2(a\,\xi_i) - J_1^2(b\,\xi_i)]}$$

$$\left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{n\, e \quad \exp.\ \left[pt - z\left\{\frac{\lambda}{\nu}\, p\left(p + \frac{1}{\lambda}\right) + \xi_i^2\right\}^{1/2}\right]}{(p^2 + n^2)}\right] dp\,.$$

The p-integral entering in (39) has been evaluated by the author elsewhere [7]. Substituting the value of the p-integral in (39) and splitting the sum

$$\sum_{i=1}^{\infty}$$

into

$$\sum_{i=1}^{k}$$

and

$$\sum_{i=k+1}^{\infty}$$

as in case (1), we get

(40) $$v = 2\omega_0 \sum_{i=1}^{\infty} \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{[J_1^2(a\,\xi_i) - J_1^2(b\,\xi_i)]}\; e \quad \exp.\left[-\left(\frac{\lambda z^2}{\nu}\right)^{1/2}\alpha\right]$$

$$\sin\left[nt - \beta\left(\frac{\lambda z^2}{\nu}\right)^{1/2}\right] + \frac{2\omega_0}{\pi}\sum_{i=1}^{k} \frac{n\, e^{-t/2\lambda}\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K\, r, a, \xi_i)}{[J_1^2(a\,\xi_i) - J_1^2(b\,\xi_i)]}$$

$$\left[\int_{-1}^{1} \frac{l\, e^{-xlt}\sin\left\{\left(\frac{\lambda}{\nu}\right)^{1/2} z\, l\,(1 - x^2)^{1/2}\right\}}{n^2 + \left(xl + \frac{1}{2\lambda}\right)^2}\, dx\right]$$

where

(41) $$l = \frac{(1 - 4\lambda\,\nu\,\xi_i^2)^{1/2}}{2\lambda},$$

(42) $$\alpha = \frac{1}{\sqrt{2}}\left[\left\{\left(\frac{\nu\,\xi_i^2}{\lambda} - n^2\right)^2 + \frac{n^2}{\lambda^2}\right\}^{1/2} + \left(\frac{\nu\,\xi_i^2}{\lambda} - n^2\right)\right]^{1/2},$$

(43) $$\beta = \frac{1}{\sqrt{2}}\left[\left\{\left(\frac{\nu\,\xi_i^2}{\lambda} - n^2\right)^2 + \frac{n^2}{\lambda^2}\right\}^{1/2} - \left(\frac{\nu\,\xi_i^2}{\lambda} - n^2\right)\right]^{1/2}$$

and

(44) $$k < \frac{1}{2\sqrt{\lambda\nu}} < k + 1.$$

Now we take the limit as $\lambda \to 0$ and obtain

(45) $$v_{vis} = \lim_{\lambda \to 0} v$$

$$= 2\omega_0 \sum_{i=1}^{\infty} \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{[J_1^2(a\,\xi_i) - J_1^2(b\,\xi_i)]}\, e\ \exp.\ \left[-\left(\frac{z^2}{2\nu}\right)^{1/2} \alpha_1\right] \sin\left[nt - \beta_1\left(\frac{z^2}{2\nu}\right)^{1/2}\right]$$

$$+ \frac{2\omega_0\, n}{\pi} \sum_{i=1}^{\infty} \frac{\{b\, J_1(a\,\xi_i) - a\, J_1(b\,\xi_i)\}\, J_1(b\,\xi_i)\, K(r, a, \xi_i)}{[J_1^2(a\,\xi_i) - J_1^2(b\,\xi_i)]} \int_0^{\infty} \frac{e\ \exp.\ [-(\nu\xi_i^2 t + yt)] \left[\sin\left(\frac{z^2 y}{\nu}\right)^{1/2}\right]}{n^2 + (y + \nu\,\xi_i^2)^2}\, dy$$

where

(46) $$\alpha_1 = [(\nu^2\,\xi_i^4 + n^2)^{1/2} + \nu\,\xi_i^2]^{1/2}$$

and

(47) $$\beta_1 = [(\nu^2\,\xi_i^4 + n^2)^{1/2} - \nu\,\xi_i^2]^{1/2}.$$

The solution givea in (40) is valid for an ordinary viscous incompressible fluid and agrees with that already obtained [3].

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ÖZET

Aynı eksenli iki silindir arasında bulunan viskoelastik bir sıvıda, tabanın ağır açısal hareketinden doğan küçük distürbansların bu sıvı içindeki yayılımı iki ayrı halde incelenmiştir.