GENERALIZATIONS OF SOME THEOREMS ON RECTILINEAR CONGRUENCES

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In this paper the definition of a Linear Normal Congruence in a sub-space Vn of a Euclidean Vm is given. Various conditions are established so that a linear congruence may be a normal congruence. The laws of refraction for a ray of the normal congruence are stated and Malus-Dupin's and Beltrami's theorems are generalized. The pitch of the pencil of a congruence as defined by Ram Behari is also generalized. Ultimately a new theorem on Union Curves is established.

1. Fundamental Formulae

Let there be a space V_n of co-ordinates x^i $(i = 1, 2, \ldots, n)$ immersed in a Euclidean V_m of co-ordinates $y^\alpha (\alpha = 1, 2, \ldots, m)$. In what follows the latin indices take the values $1, 2, \ldots, n$ and greek indices the values $1, 2, \ldots, m$. Suppose the metrics of V_n and V_m are positive definite and are respectively defined by $g_{ij}\, dx^i\, dx^j$ and $\delta_{\alpha\beta}\, dy^\alpha\, dy^\beta$ where $\delta_{\alpha\beta}$ are Kronecker deltas:

$$\delta_{\alpha\beta} = 1 \qquad \text{when } \alpha = \beta,$$

$$= 0 \qquad \text{when } \alpha \neq \beta.$$

We have then the relation

$$g_{ij} = \delta_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j} \tag{1.1}$$

where a comma (,) followed by latin indices denotes the generalized covariant derivative or the tensor derivative with respect to the x's based on g_{ij} [8, p. 51]. Let $N^\alpha{}_{\nu|}$ be the contravariant components in V_m of a system of $(m\text{-}n)$ mutually orthogonal unit vectors normal to V_n; then the following relations are satisfied (8):

$$\delta_{\alpha\beta}\, N^\alpha{}_{\nu|}\, N^\beta{}_{\mu|} = \delta^\mu{}_\nu, \tag{1.2}$$

$$\delta_{\alpha\beta}\, y^\alpha{}_{,i}\, N^\beta{}_{\nu|} = 0. \tag{1.3}$$

Suppose $A_{\nu|ij}$ are the components in V_n of a symmetric covariant tensor of the secont order corresponding to a system $N^{\alpha}{}_{\nu|}$ of the normals of V_n then [8, p. 163]

$$A_{\nu|ij} = \delta_{\alpha\beta}\, y^{\alpha}{}_{,ij}\, N^{\beta}{}_{\nu|}\,. \tag{1.4}$$

We also have the Gauss formulae for a sub-space V_n of V_m [8, p. 163]

$$y^{\alpha}{}_{,ij} = \sum_{\nu} A_{\nu|ij}\, N^{\alpha}{}_{\nu|}\,. \tag{1.5}$$

2. Linear Congruences

Let us consider a set of $(m\text{-}n)$ linear congruences which are such that through each point of a sub-space V_n of V_m, a curve of each congruence passes.

A normal congruence in Euclidean space is defined as a congruence which is capable of orthogonal intersection by a sub-space and hence a family of suhspaces.

The equation of the line of the congruence given by

$$Y^{\alpha} = y^{\alpha} + t\,\lambda^{\alpha}{}_{\tau|} \tag{2.1}$$

where Y^{α} is a point on the line distant t from y^{α} and $\lambda^{\alpha}{}_{\tau|}$ are the unit tangents in the direction of the line of the congruence $\lambda_{\tau|}$. Differentiating we have

$$dY^{\alpha} = dY^{\alpha} + td\,(\lambda^{\alpha}{}_{\tau|}) + \lambda^{\alpha}{}_{\tau|}\, dt. \tag{2.2}$$

Hence in order that the congruence $\lambda_{\tau|}$ be normal we must have

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\, dY^{\beta} = 0.$$

Multiplying both sides of (2.2) by $\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}$ we have

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\, dy^{\beta} + t\,\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\, d\,(\lambda^{\beta}{}_{\tau|}) + \delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\,\lambda^{\beta}{}_{\tau|}\, dt = 0 \tag{2.3}$$

since $\lambda^{\alpha}{}_{\tau|}$ is a unit vector, therefore

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\,\lambda^{\beta}{}_{\tau|} = 1 \tag{2.4}$$

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\, d(\lambda^{\beta}{}_{\alpha|}) = 0.$$

But in a Euclidean space the intrinsic derivative is equal to the total derivative. Therefore using (2.4), (2.3) becomes

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|}\, dy^{\beta} = -\,dt.$$

Since this holds for all values of the differential dx^{i} we have

(2.5) $$\delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|}\ y^{\beta}{}_{,i} = -t_{,i}.$$

Similarly

(2.6) $$\delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|}\ y^{\beta}{}_{,j} = -t_{,j}$$

covariant differentiation of (2.5) and (2.6) yields

$$\delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|,j}\ y^{\beta}{}_{,i} + \delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|}\ y^{\beta}{}_{,ij} = -t_{,ij}\,,$$
$$\delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|,i}\ y^{\beta}{}_{,j} + \delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|}\ y^{\beta}{}_{,ji} = -t_{,ji}\,.$$

Since $y^{\beta}{}_{,ij}$ and $t_{,ij}$ are both symmetric in co-variant indices, we have the condition of normality as

(2.7) $$\delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|,i}\ y^{\beta}{}_{,j} = \delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|,j}\ y^{\beta}{}_{,i}.$$

$\lambda^{\alpha}{}_{\tau|}$ being a vector in m dimensional Euclidean space can be expressed as a linear combination of m vectors which do not lie in the same geodesic surface. Let those m vectors be the n vectors $y^{\alpha}{}_{,i}$ and the $(m\text{-}n)$ vectors $N^{\alpha}{}_{\nu|}$. Hence

(2.8) $$\lambda^{\alpha}{}_{\tau|} = t^{l}{}_{\tau|}\ y^{\alpha}{}_{,l} + \sum_{\nu} C_{\nu\tau|}\ N^{\alpha}{}_{\nu|}\,,$$

where if $\vartheta_{\nu\tau|}$ is the angle between $\lambda^{\alpha}{}_{\tau|}$ and $N^{\alpha}{}_{\nu|}$, then

(2.9) $$C_{\nu\tau|} = \delta_{\alpha\beta}\ \lambda^{\alpha}{}_{\tau|}\ N^{\beta}{}_{\nu|} = \text{Cos}\ \vartheta_{\nu\tau|}\,,$$

(2.10) $$1 - \sum_{\nu} \text{Cos}^2\ \vartheta_{\nu\tau|} = g_{ij}\ t^{i}{}_{\tau|}\ t^{j}{}_{\nu|}\,.$$

Covariant differentiation of (2.8) yields

(2.11) $$\lambda^{\alpha}{}_{\tau|,i} = t^{l}{}_{\tau|,i}\ y^{\alpha}{}_{,l} + t^{l}{}_{\tau|}\ y^{\alpha}{}_{,li} + \sum_{\nu} C_{\nu\tau|,i}\ N^{\alpha}{}_{\nu|} + \sum_{\nu} C_{\nu\tau|}\ N^{\alpha}{}_{\nu|,i}$$

But

$$N^{\alpha}{}_{\nu|,i} = -A_{\nu|ik}\ g^{kl}\ y^{\alpha}{}_{,l} + \sum_{\mu} \vartheta_{\mu\nu|i}\ N^{\alpha}{}_{\nu|}$$

where

$$\vartheta_{\mu\nu|i} = \delta_{\alpha\beta}\ N^{\alpha}{}_{\nu|,i}\ N^{\beta}{}_{\mu|}\,.$$

Hence (2.11) becomes

(2.12) $$\lambda^{\alpha}{}_{\tau|,i} = q^{l}{}_{\tau|i}\ y^{\alpha}{}_{,l} + \sum_{\nu} r_{\nu\tau|i}\ N^{\alpha}{}_{\nu|}$$

where

(2.13) $$q^{l}_{\tau|i} = t^{l}_{\tau|,i} - \sum_{\nu} C_{\nu\tau|} A_{\nu|ik}\, g^{kl}$$

and

(2.14) $$\nu_{\nu\tau|i} = t^{l}_{\tau|} A_{\nu|li} + C_{\nu\tau|,i} + \sum_{\mu} C_{\mu\tau|} \vartheta_{\mu\nu|i}.$$

Hence we deduce from (2.12)

$$\delta_{\alpha\beta}\, \lambda^{\alpha}_{\tau|,i}\, y^{\beta}_{,j} = q^{l}_{\tau|i}\, g_{lj}$$

(2.15) $$= q_{\tau|ij}.$$

Hence from (2.7) and (2.15) it follows that in order that the congruence may be normal, $q_{\tau|ij}$ should be symmetric in the indices i and j. From (2.13) we have

$$q^{l}_{\tau|i}\, g_{lj} = t_{\tau}{}^{l}{}_{|,i}\, g_{lj} - \sum_{\nu} C_{\nu\tau|} A_{\nu|ik}\, g^{kl}\, g_{lj}$$

$$= t^{l}_{\tau|,i}\, g_{lj} - \sum_{\nu} C_{\nu\tau|} A_{\nu|ik}\, \delta^{k}_{j}$$

or

(2.16) $$q_{\tau|ij} = t_{\tau|i,j} - \sum_{\nu} C_{\nu\tau|} A_{\nu|ij}.$$

Similarly

(2.17) $$q_{\tau|ji} = t_{\tau|i,j} - \sum_{\nu} C_{\nu\tau|} A_{\nu|ji}.$$

Since for a normal congruence $q_{\tau|ij}$ is symmetric, then

(2.18) $$t_{\tau|ji} - t_{\tau|i,j} = \sum_{\nu} C_{\nu\tau|} (A_{\nu|ij} - A_{\nu|ji}).$$

This gave us

(2.19) $$t_{\tau|j,i} = t_{\tau|i,j}$$

since $A_{\nu|ij}$ is symmetric in the indices i and j. Hence: another condition for the congruence $\lambda_{\tau|}$ to be a normal congruence is that $t_{\tau|i,j}$ be symmetric.

3. Malus-Dupin's Theorem

Let the curve $\lambda_{\tau|}$ be refracted through the sub-space V_n to the curve $\bar{\lambda}_{\tau|}$. We call $\lambda^{\alpha}_{\tau|}$ the incident ray, $\bar{\lambda}^{\alpha}_{\tau|}$ the refracted ray and V_n the space of refraction. Let $\vartheta_{\nu\tau|}$ be the angle between $\lambda^{\alpha}_{\tau|}$ and $N^{\alpha}_{\nu|}$. Before proceeding to generalize this theorem we shall state the following laws of refraction for a ray of a congruence in a Euclidean V_m. Let us assume that the following two laws hold for refraction :

(*i*) The incident ray, the set of normals and the refracted ray lie in the same goedesic surface.

(*ii*) The ratio

$$\frac{1-\sum_{\nu}\text{Cos}^2\,\vartheta_{\nu\tau|}}{1-\sum_{\nu}\text{Cos}^2\,\bar{\vartheta}_{\nu\tau|}} = \text{const} = M^2\,(\text{say}). \tag{3.1}$$

If the incident ray, the set of normals and the refracted ray lie in the same geodesic surface, then we have

$$\lambda^{\alpha}_{\tau|} = \sum_{\nu} L_{\tau\nu|}\,N^{\alpha}_{\tau|} + M\,\bar{\lambda}^{\alpha}_{\tau|}\,. \tag{3.2}$$

Multiplying both sides of (3.2) by $\delta_{\alpha\beta}\,N^{\beta}_{\nu|}$ and summing with respect to α we have

$$\text{Cos}\,\vartheta_{\nu\tau|} = L_{\tau\nu|} + M\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|}\,.$$

Hence the equation (3.2) assumes the form

$$\lambda^{\alpha}_{\tau|} = \sum_{\nu}(\text{Cos}\,\vartheta_{\nu\tau|} - M\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|})\,N^{\alpha}_{\nu|} + M\,\bar{\lambda}^{\alpha}_{\tau|}$$

or

$$\lambda^{\alpha}_{\tau|} - \sum_{\nu}(\text{Cos}\,\vartheta_{\nu\tau|} - M\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|})\,N^{\alpha}_{\nu|} = M\,\lambda^{\alpha}_{\tau|}\,. \tag{3.3}$$

Taking the magnitude of the vectors of both the sides of this equation we have

$$1 + \Big[\sum_{\nu}(\text{Cos}\,\vartheta_{\nu\tau|} - M\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|})\Big]^2 - 2\sum_{\nu}(\text{Cos}\,\vartheta_{\nu\tau|} - M\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|})\,\text{Cos}\,\bar{\vartheta}_{\nu\tau|} = M^2$$

or

$$1+\sum_{\nu}\operatorname{Cos}^2\vartheta_{\nu\tau|}+M^2\sum_{\nu}\operatorname{Cos}^2\bar{\vartheta}_{\nu\tau|}-2M\sum_{\nu}\operatorname{Cos}\vartheta_{\nu\tau|}\operatorname{Cos}\bar{\vartheta}_{\nu\tau|}-2\sum_{\nu}\operatorname{Cos}^2\bar{\vartheta}_{\nu\tau|}$$
$$+2M\sum_{\nu}\operatorname{Cos}\vartheta_{\nu\tau|}\operatorname{Cos}\bar{\vartheta}_{\nu\tau|}=M^2$$

or

$$\frac{1-\sum_{\nu}\operatorname{Cos}^2\vartheta_{\nu\tau|}}{1-\sum_{\nu}\operatorname{Cos}^2\bar{\vartheta}_{\nu\tau|}}=M^2.$$

Hence by the second law of refraction, M must be constant.

Now

$$\lambda^{\alpha}{}_{\tau|}=\sum_{\nu}L_{\tau\nu|}N^{\alpha}{}_{\nu|}+M\bar{\lambda}^{\alpha}{}_{\tau|}.$$

Differentiating co-variantly we have

$$\lambda^{\alpha}{}_{\tau|,i}=\sum_{\nu}L_{\tau\nu|,i}N^{\alpha}{}_{\nu|}+\sum_{\nu}L_{\tau\nu|}N^{\alpha}{}_{\nu|,i}+M\bar{\lambda}^{\alpha}{}_{\tau|,i}. \tag{3.4}$$

Multiplying both sides of (3.4) by $\delta_{\alpha\beta}\,y^{\beta}{}_{,j}$ we have

$$\delta_{\alpha\beta}\lambda^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}=\sum_{\nu}L_{\tau\nu|,i}\,\delta_{\alpha\beta}N^{\alpha}{}_{\nu|}y^{\beta}{}_{,j}+\sum_{\nu}L_{\tau\nu|}\,\delta_{\alpha\beta}N^{\alpha}{}_{\nu|,i}\,y^{\beta}{}_{,j}$$
$$+M\,\delta_{\alpha\beta}\bar{\lambda}^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}$$

$$=\sum_{\nu}L_{\tau\nu|,i}A_{\nu|ij}+M\,\delta_{\alpha\beta}\bar{\lambda}^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}. \tag{3.5}$$

Similarly we have

$$\delta_{\alpha\beta}\lambda^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{i}=\sum_{\nu}L_{\tau\nu|,j}A_{\nu|ji}+M\,\delta_{\alpha\beta}\bar{\lambda}^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{,i}. \tag{3.6}$$

Subtracting (3.6) from (3.5) we have

$$\delta_{\alpha\beta}\lambda^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}-\delta_{\alpha\beta}\lambda^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{,i}=\sum_{\nu}L_{\tau\nu|}(A_{\nu|ij}-A_{\nu|ji})$$
$$+M(\delta_{\alpha\beta}\bar{\lambda}^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}-\delta_{\alpha\beta}\bar{\lambda}^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{,i}).$$

Now for a normal congruence $\delta_{\alpha\beta}\lambda^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}$ is symmetric, therefore the above equation takes the form

$$M\left(\delta_{\alpha\beta}\,\bar{\lambda}^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}-\delta_{\alpha\beta}\,\bar{\lambda}^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{,i}\right)=0$$

But we have proved that M is a constant, hence

$$\delta_{\alpha\beta}\,\bar{\lambda}^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,j}-\delta_{\alpha\beta}\,\bar{\lambda}^{\alpha}{}_{\tau|,j}\,y^{\beta}{}_{,i}=0 \tag{3.7}$$

or

$$\delta_{\alpha\beta}\,\lambda^{\alpha}{}_{\tau|,i}\,y^{\beta}{}_{,} \text{ is symmetric.}$$

Hence:

If the incident rays $\lambda^{\alpha}{}_{\tau|}$ form a normal congruence, then the refracted rays $\bar{\lambda}^{\alpha}{}_{\tau|}$ also form a normal congruence.

4. Beltrami's Theorem

Let the space of reference V_n be deformed in such a manner that the direction of the normal congruence $\lambda_{\tau|}$ with respect to the space remain unaltered. Then

$$\lambda^{\alpha}{}_{\tau|}=\bar{t}^{l}{}_{\tau|}\,\bar{y}^{\alpha}{}_{,l}+\sum_{\nu}C_{\nu\tau|}\,\bar{N}^{\alpha}{}_{\nu|} \tag{4.1}$$

and

$$1-\sum_{\nu}\operatorname{Cos}^2\vartheta_{\nu\tau|}=t^{l}{}_{\tau|}\,t_{\tau|l}=\bar{t}^{l}{}_{\tau|}\,\bar{t}_{\tau|l}$$

$$-\frac{\partial}{\partial x^{i}}\left(\sum_{\nu}\operatorname{Cos}^2\vartheta_{\nu\tau|}\right)=2\,t^{l}{}_{\tau|,i}\,t_{\tau|l}=2\,\bar{t}^{l}{}_{\tau|,i}\,\bar{t}_{\tau|l}$$

so that we have

$$t^{l}{}_{\tau|,i}\,t_{\tau|l}=\bar{t}^{l}{}_{\tau|,i}\,\bar{t}_{\tau|l}$$

or

$$t_{\tau|m,i}\,t_{\tau|l}\,g^{lm}=\bar{t}_{\tau|m,i}\,\bar{t}_{\tau|l}\,\bar{g}^{lm}. \tag{4.2}$$

Hence if $t_{\tau|m,i}$ is symmetric, then $\bar{t}_{\tau|m,i}$ is also symmetric and this is the condition for the congruence $\lambda_{\tau|}$ to be normal.

Hence:

If the space of reference V_n of a normal congruence be deformed in such a manner that the direction of the congruence with respect to the space remain unaltered, the congruence continues to he normal.

5. Let $\lambda^{\alpha}{}_{\tau|}$ he the incident ray, $\bar{\lambda}^{\alpha}{}_{\tau|}$ the refracted ray and $N^{i}{}_{\nu|}$ the components of a set of unit normals at a point P of V_n.

Then we have as in MALUS-DUPIN's theorem

(5.1) $$\lambda^{\alpha}{}_{\tau|} = \sum_{\nu} L_{\tau\nu|} N^{\alpha}{}_{\nu|} + M \bar{\lambda}^{\alpha}{}_{\tau|}.$$

Multiplying both sides of (5.1) by $\delta_{\alpha\beta}\, y^{\beta}{}_{,i}$ we have

$$\delta_{\alpha\beta}\, \lambda^{\alpha}{}_{\tau|}\, y^{\beta}{}_{,i} = \sum_{\nu} L_{\tau\nu|}\, \delta_{\alpha\beta}\, N^{\alpha}{}_{\nu|}\, y^{\beta}{}_{,i} + M\, \delta_{\alpha\beta}\, \bar{\lambda}^{\alpha}{}_{\tau|}\, y^{\beta}{}_{,i}.$$

Using (2.8) this becomes

$$\delta_{\alpha\beta}\, t^{l}{}_{\tau|}\, y^{\beta}{}_{,i}\, y^{\alpha}{}_{,l} = M\, \delta_{\alpha\beta}\, \bar{t}^{l}{}_{\tau|}\, y^{\beta}{}_{,i}\, y^{\alpha}{}_{,l}$$

or

$$t^{l}{}_{\tau|}\, g_{il} = M\, \bar{t}^{l}{}_{\tau|}\, g_{il}$$

or

(5.2) $$t_{\tau|i} = M\, \bar{t}_{\tau|i}$$

In euclidean 3-space if C is a closed curve the pitch of the pencil of the congruence

$$\lambda^{i} = p^{\alpha}\, x^{i}{}_{,\alpha} + qX^{i}$$

is given by

(5.3) $$p = \int_{c} \lambda^{i}\, dx^{i} = \int_{c} (p^{\alpha}\, x^{i}{}_{,\alpha} + qX^{i})\, x^{i}{}_{,\delta}\, du^{\delta} = \int_{c} p_{\delta}\, du^{\delta}$$

where C is a closed curve.

Similarly let us consider a closed curve in V_n, then we define the pitch of the congruence by

$$p = \int_{c} \delta_{\alpha\beta}\, \lambda^{\alpha}{}_{\tau|}\, dy^{\beta}$$

$$= \int_{c} \delta_{\alpha\beta}\, [t^{j}{}_{\tau|}\, y^{\alpha}{}_{,j} + \sum_{\nu} C_{\nu\tau|}\, N^{\alpha}{}_{\nu|}]\, y^{\beta}{}_{,i}\, dx^{i}$$

$$= \int_{c} \delta_{\alpha\beta}\, t^{j}{}_{\tau|}\, y^{\alpha}{}_{,j}\, y^{\beta}{}_{,i} + \sum_{\nu} C_{\nu\tau|}\, \delta_{\alpha\beta}\, N^{\alpha}{}_{\nu|}\, y^{\beta}{}_{,i}]\, dx^{i}$$

$$= \int_{c} g_{ij}\, t^{j}{}_{\tau|}\, dx^{i}$$

(5.4) $$= \int_{c} t_{\tau|i}\, dx^{i}.$$

Similarly if p' is the pitch of the refracted ray then

$$p = \int_c t_{\tau|i}\, dx^i = \int_c M \bar{t}_{\tau|i}\, dx^i$$

$$= M \int_c \bar{t}_{\tau|i}\, dx^i$$

$$= M\bar{p}.$$

Hence :

The pitch of the pencil of ray of the congruence formed by the incident rays remains unaltered by refraction except for the constant of refraction between the two media.

6. Union Curves

A union curve C corresponding to a curve $\lambda_{\tau|}$ of the congruence of a sub-space V_n of V_m is defined as a curve which has the property that at point P of the curve the osculating geodesic surface contains the tangent $(\lambda^\alpha_{\tau|})$ to the curve of the congruence through the point P. Hence for a union curve we have

$$\lambda^\alpha_{\tau|} = \nu_{\tau|} \frac{dy^\alpha}{ds} + s_{\tau|} q^\alpha \tag{6.1}$$

where $\frac{dy^\alpha}{ds}$ are the components of the tangent vector to the curve and q^α are the contravariant components of the curvature vectore of G relative to V_m are given by [4, p. 3]

$$q^\alpha = \Sigma\, A_{\nu|ij} \frac{dx^i}{ds} \frac{dx^j}{ds} N^\alpha_{\nu|} + g^\alpha{}_{,i}\, p^i, \tag{6.2}$$

p^i being the contravariant components of the curvature vector of C relative to V_n.

Let $\varphi_{\tau|}$ be the angle between $\frac{dy^\alpha}{ds}$ and $\lambda^\alpha_{\tau|}$. Then

$$\delta_{\alpha\beta} \frac{dy^\alpha}{ds} \lambda^\beta_{\tau|} = \text{Cos}\, \varphi_{\tau|}. \tag{6.3}$$

Multiplying both sides of (6.1) by $\delta_{\alpha\beta} \frac{dy^\beta}{ds}$ we have

$$\text{Cos}\, \varphi_{\tau|} = \nu_{\tau|}. \tag{6.4}$$

Again multiplying both sides of (6.1) by $\delta_{\alpha\beta}$ we have

$$1 = \nu_{\tau|} \operatorname{Cos} \varphi_{\tau|} + s_{\tau|}\, \delta_{\alpha\beta}\, \lambda^{\beta}_{\tau|}\, q^{\alpha},$$

or

(6.5) $$\operatorname{Sin}^2 \varphi_{\tau|} = s_{\tau|}\, \delta_{\alpha\beta}\, \lambda^{\beta}_{\tau|}\, q^{\alpha}.$$

Equation (6.1) can be written as

$$\lambda^{\alpha}_{\tau|} - s_{\tau|}\, q^{\alpha} = \nu_{\tau|} \frac{dy^{\alpha}}{ds}.$$

Taking squares of the magnitudes of the vectors of this equation we have

(6.6) $$1 + s^2_{\tau|}\, k^2 - 2 s_{\tau|}\, \delta_{\alpha\beta}\, \lambda^{\beta}_{\tau|}\, q^{\alpha} = \cos^2 \varphi_{\tau|}$$

where k is the curvature of the curve C. Equation (6.6) yields on simplification

$$k^2\, s^2_{\tau|} = \operatorname{Sin}^2 \varphi_{\tau|},$$

or

(6.7) $$s_{\tau|} = \frac{\operatorname{Sin} \varphi_{\tau|}}{k}.$$

Therefore equation (6.1) becomes

$$\lambda^{\alpha}_{\tau|} = \operatorname{Cos} \varphi_{\tau|} \frac{dy^{\alpha}}{ds} + \frac{\operatorname{Sin} \varphi_{\tau|}}{k} q^{\alpha}$$

or

(6.8) $$q^{\alpha} = -k \operatorname{Cot} \varphi_{\tau|} \frac{dy^{\alpha}}{ds} + \frac{k}{\operatorname{Sin} \varphi_{\tau|}} \lambda^{\alpha}_{\tau|}.$$

Using (6.2) the equation (6.8) becomes

(6.9) $$\sum_{\nu} A_{\nu|ij} \frac{dx^i}{ds} \frac{dx^j}{ds} N^{\sigma}_{\nu|} + y^{\alpha}_{,i}\, p^i = -k \operatorname{Cot} \varphi_{\tau|} \frac{dy^{\alpha}}{ds} + \frac{k}{\operatorname{Sin} \varphi_{\tau|}} \lambda^{\alpha}_{\tau|}.$$

Multiplying both sides of (6.9) by ${}_{\alpha\beta}N^{\beta}_{\nu|}$ and summing up we have

(6.10) $$\sum_{\nu} A_{\nu|ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = k \operatorname{Cosec} \varphi_{\tau|} \operatorname{Cos} \vartheta_{\nu\tau|}.$$

Squaring (6.10) we have a simplificative

$$k_n^2 = k^2 \operatorname{Cosec}^2 \varphi_{\tau|} \sum_{\nu} \operatorname{Cos}^2 \vartheta_{\nu\tau|},$$

or

$$k_n = k \operatorname{Cosec} \varphi_{\tau|} \sqrt{1 - t^j_{\tau|}\, t_{\tau|j}}.$$

Hence :

If the Union curve is an asymptotic curve, then the tangent geodesic surface to V_n at P coincides with the osculating geodesic surface to the curve or k may be zero, that is the Union curve can be a geodesic also if the Union curve is an asymptotic line.

Multiplying (6.9) by $\delta_{\alpha\beta}\ \lambda^{\beta}_{\tau|}$ we have

$$\sum_{\nu} A_{\nu|ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\delta_{\alpha\beta}\ N^{\alpha}_{\nu|}\ \lambda^{\beta}_{\tau|}+\delta_{\alpha\beta}\ \lambda^{\beta}_{\tau|}\ y^{\alpha}_{,i}\,p^i$$

$$=-k\,\mathrm{Cot}\,\varphi_{\tau|}\ \delta_{\alpha\beta}\ \lambda^{\beta}_{\tau|}\frac{dy^{\alpha}}{ds}+\frac{k}{\mathrm{Sin}\,\varphi_{\tau|}}\delta_{\alpha\beta}\ \lambda^{\alpha}_{\tau|}\ \lambda^{\beta}_{\tau|}\ ,$$

or

$$\sum_{\nu} A_{\nu|ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\mathrm{Cos}\,\vartheta_{\nu\tau|}+t_{\tau|i}\ p^i=-k\,\mathrm{Cot}\,\varphi_{\tau|}\,\mathrm{Cos}\,\varphi_{\tau|}+\frac{k}{\mathrm{Sin}\,\varphi_{\tau|}}=k\,\mathrm{Sin}\,\varphi_{\tau|}\,,$$

or

$$\sum_{\nu} k\,\mathrm{Cosec}\,\varphi_{\tau|i}\ \mathrm{Cos}^2\,\vartheta_{\nu\tau|}+t_{\tau|i}\ p^i=k\,\mathrm{Sin}\,\varphi_{\tau|}\ ,$$

or

$$k=\frac{t_{\tau|i}\ p^i\,\mathrm{Sin}\,\varphi_{\tau|}}{\mathrm{Sin}^2\,\varphi_{\tau|}-\sum_{\nu}\mathrm{Cos}^2\,\vartheta_{\nu\tau|}}\,.$$

REFERENCES

[1] BEHARI, R. : *Generalizations of the theorems of Malus - Dupin, Beltrami and Ribaucour in Rectilinear Congruences.* Jour. Ind. Math. Soc. New Series Vol II, **2**, 45—60 (1936).

[2] BEHARI, R. : *Differential Geometry of Ruled Surfaces.* Lucknow University Studies. No XVIII (1946).

[3] BEHARI, R. AND MISHRA R. S. : *Some Formulae in Rectilinear Congruences.* Proc. Nat. Inst. Sci. India XV, **8**, (1949).

[4] MISHRA, R. S. : *Sur Certaines Courbes Appartenant à un Sous Espace d'un Espace Riemannien.* Bull. Des Sci. Math. Second Series, **19**, (1050).

[5] MISHRA R. S. : *Union Curves and hyper asymptotic curves on the surface of reference of a Rectilinear Congruence.* Bull. Cal. Math. Soc., **42**, 213—16 (1950).

[6] MCCONNEL : **Applications of the Absolute Differential Calculus,** BLACKIE AND SON, London (1931).

[7] SPRINGER : *Union curves of a hypersurface.* Canadian Jour. Math., 00457—4 (1950).

[8] WEATHERBURN : **An Introduction to the Riemannian Geometry,** Cambridge (1050).

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ÖZET

Bu yazıda bir Öklid Vm m-boyutlu uzayına daldırılmış bulunan Vn alt-uzayında normal doğru kongrüansının tarifi verilmiş, ve bir kongrüansın normal olması için gerçeklenmesi gereken şartlar elde edilmiştir. Ayrıca bir normal kongrüansın ışınlarının yansıma kanunları ifade edildikten sonra MALUS-DUPIN ve BELTRAMI teoremlerinin teşmili yapılmıştır. RAM BEHARI tarafından üç boyutlu uzaydaki bir kongrüansın bir ışın demeti için tarif edilen «pitch» kavramı teşmil edilmiş ve nihayet «birleşim eğrileri» için yeni bir netice ispat olunmuştur.