ON LINDELOF'S PROXIMATE ORDER

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The object of this paper is to prove some inequalities for the upper and lower limits of certain ratios of particular integral functions of LINDELÖF's proximate order.

Introduction : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function having order $\varrho\,(0 < \varrho < \infty)$ and LINDELÖF's proximate order $\varrho(r)$ (see [1], p. 54). Define :

$$M(r) = \exp \int_{\delta}^{r} \frac{\varrho(x)}{x}\, dx \; ; \qquad N(r) = \int_{\delta}^{r} \frac{n(x)\,\varrho(x)}{x}\, dx \; ; \; \delta > 0,$$

where $n(x)$ is a non-decreasing function of x, at least for $x \geq x_0$. Let

$$\overline{\underline{\lim}}_{r \to \infty} \frac{N(r)}{M(r)} = \frac{A}{D} \; ; \qquad \overline{\underline{\lim}}_{r \to \infty} \frac{n(r)}{M(r)} = \frac{C}{D} \; ;$$

The following relationships between these limits may then be proved.

Theorem :

(*i*) $A \leq C$; (*ii*) $B \leq D \left\{1 + \log (C/D)\right\}$,

(*iii*) $A \geq \frac{C}{e}\, e^{D/C}$; (*iv*) $B \geq D$; (*iv'*) $A \geq D$;

(*v*) $C \leq Ae$; (*vi*) $D + C \leq Ae$.

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Corollary: *Equality cannot hold at the same time in (iv′) and (vi).*

To prove the theorem, the following intermediate lemma is required:

Lemma: *If $M(r)$ is defined as above, then for every finite $\eta \geq 0$,*

$$\frac{M(r+\eta r)}{M(r)} \to (1+\eta)^{\varrho}$$

uniformly as $r \to \infty$.

Proof of the lemma: We have

$$\log\left\{\frac{M(r+\eta r)}{M(r)}\right\} = \int_r^{r+\eta r} \frac{\varrho(x)}{x}\,dx = \Big[\varrho(x)\log x\Big]_{x=r}^{x=r+\eta r} - \int_r^{r+\eta r} \varrho'(x)\log x\,dx.$$

But

$$\left|\int_r^{r+\eta r} \varrho'(x)\log x\,dx\right| < \varepsilon \log(1+\eta),\ r > r_0(\varepsilon),$$

by (*iii*), ([1], p. 54). Hence for sufficiently large r

$$\log\left\{\frac{M(r+\eta r)}{M(r)}\right\} = \log\frac{(1+\eta)\,r)^{\varrho(r+\eta r)}}{r^{\varrho(r)}}\Bigg\} + O(1) \to \log(1+\eta)^{\varrho},$$

by lemma 1 ([1], p. 55), uniformly as $r \to \infty$.

Proof of the theorem: We have

$$N(r+\eta r) = O(1) + \int_{r_0}^{r} \frac{n(x)\,\varrho(x)}{x}\,dx + \int_r^{r+\eta r} \frac{n(x)\,\varrho(x)}{x}\,dx$$

$$= O(1) + \int_{r_0}^{r} \frac{n(x)\,M'(x)}{M(x)}\,dx + \int_r^{r+\eta r} \frac{n(x)\,M'(x)}{M(x)}\,dx$$

$$< O(1) + (C+\varepsilon)\,M(r) + n(r+\eta r)\log\left\{\frac{M(r+\eta r)}{M(r)}\right\}.$$

Therefore

$$\frac{M(r+\eta r)}{N(r+\eta r)} < o(1) + (C+\varepsilon)\,\frac{M(r)}{M(r+\eta r)} + \frac{n(r+\eta r)}{M(r+\eta r)}\log\left\{\frac{M(r+\eta r)}{M(r)}\right\}.$$

Hence

(1) $$A \leq \frac{C}{(1+\eta)^{\varrho}} + \varrho\,C\log(1+\eta);$$

(2) $$B \leq \frac{C}{(1+\eta)^{\varrho}} + \varrho D \log(1+\eta).$$

Substituting $\eta = (C/D)^{1/\varrho} - 1$ in (2)] and $\eta = 0$ in (1), we get (*ii*) and (*i*) respectively. Similarly we have

$$\frac{N(r+\eta r)}{M(r+\eta r)} > o(1) + (D-\varepsilon)\frac{M(r)}{M(r+\eta r)} + \frac{n(r)}{M(r)} \cdot \frac{M(r)}{M(r+\eta r)} \log\left\{\frac{M(r+\eta r)}{M(r)}\right\}.$$

Therefore

(3) $$A \geq \frac{D}{(1+\eta)^{\varrho}} + \frac{C\varrho}{(1+\eta)^{\varrho}} \log(1+\eta);$$

(4) $$B \geq \frac{D}{(1+\eta)^{\varrho}} + \frac{D\varrho}{(1+\eta)^{\varrho}} \log(1+\eta).$$

Substituting $\eta = \exp\{(C-D)/\varrho C\} - 1$ in (3) and $\eta = 0$ in (4); (*iii*) and (*iv*) are obtained respectively.

Now from (*iii*)

(5) $$Ae \geq C(1 + D/C + \cdots) \geq C,$$

and so (*v*) follows. Also from (5)

$$Ae \geq C(1 + D/C),$$

and so (*vi*) follows. (*iv'*) is obvious from (*iv*).

Proof of the corrollary: Suppose first $A = D$. Then from (3), for η sufficiently small

$$C \leq \frac{\{(1+\eta)^{\varrho} - 1\} A}{\varrho \operatorname{iog}(1+\eta)} = \left\{\frac{\varrho\eta + O(\eta^2)}{\varrho\eta + O(\eta^2)}\right\} A \to A,$$

as $\eta \to 0$. Hence

$$C \leq A = D,$$

but $C \geq D$ always, hence $C = D = A$. Therefore

$$C + D = 2A < eA.$$

Next suppose $C + D = eA$, then we say that $D < A$, for if it were equal to A, then from the above argument $C + D < eA$, contrary to the hypothesis.

Remark: The above results include those of S. K. Singh [2].

REFERENCES

[1] CARTWRIGHT, M. L. : **Integral functions,** CAMBRIDGE UNIVERSITY PRESS, Cambridge (1956).

[2] SINGH, S. K. : *On the maximum term and its rank of an entire function, Acta Mathematica,* 94, 1—11, (1955).

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ÖZET

Bu yazıda LINDELÖF «yaklaşık mertebesinin» bazı özel integral fonksiyonlariyle meydana getirilen birtakım kesirlerinin alt ve üst limitleri arasında mevcut birkaç eşitsizlik ispat edilmektedir.