A THEOREM ON STEP FUNCTIONS (II)

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The object of this paper is to give a criterion for the convergence or divergence of an infinite series associated to a step function by means of an integral containing characteristic quantities of the function itself.

1. Let

$$f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \; (s=\sigma+it),$$

where the sequence $\{\lambda_n\}$ is chosen in such a manner so as to satisfy the following conditions :

(*i*) $$0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty\,;$$

(*ii*) $$\overline{\lim_{n\to\infty}}\frac{n}{\lambda_n}=\frac{D}{d}\,; \quad (0<d\leqq D<\infty),$$

(*iii*) $$\overline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)=\frac{m}{h}; \quad h>0\,; \quad hD\leqq 1,$$

be an entire function. Let $M(\sigma)$, $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ stand as usual. Suppose χ_n denotes the «Sprungstellen» for $f(s)$, equivalent to $\log|a_n/a_{n-1}|/(\lambda_n-\lambda_{n-1})$ (for properties of χ_n see [1], p. 718). Since $\lambda_{\nu(\sigma)}$ is a step function with jumps at χ_n, we have ([2], p. 43)

$$\lambda_{\nu(\sigma)}=\sum_{\chi_n\leqq\sigma}(\lambda_n-\lambda_{n-1}), \quad (\lambda_0=\lambda_{-1}).$$

Throughout this note we suppose, as we may without loss of generality, that $\mu(0)=1$. I wish to prove the following:

2. Theorem: *Let $f(s)$ be an entire function of finite order $(R)>0$; then the convergence or divergence of the integral:*

$$\int_0^\infty \frac{\log M(\sigma)}{e^{\alpha\sigma}}\, d\sigma \tag{2.1}$$

implies the convergence or divergence of the infinite series:

$$\sum^{\infty} |a_n|^{\alpha/\lambda_n}. \tag{2.2}$$

To prove the result, the following necessary lemmas are required:

Lemma 1: *Let $\Psi'(x)$ be any function integrable for $x > 0$, then*

$$\sum_{\chi_n \leqq \sigma} (\lambda_n - \lambda_{n-1})\, \Psi(\chi_n) = \Psi(\sigma)\, \lambda_{\nu(\sigma)} - \int_0^\sigma \lambda_{\nu(t)}\, \Psi'(t)\, dt.$$

For the proof, sec [8].

Lemma 2: *The series:*

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1})\, e^{-\alpha\chi_n} \tag{3.1}$$

and the integral:

$$\int_0^\infty \frac{\lambda_{\nu(t)}}{e^{\alpha t}}\, dt$$

converge or diverge together.

For, by lemma 1, we have

$$\sum_{n=1}^{N} (\lambda_n - \lambda_{n-1})\, e^{-\alpha\chi_n} = \frac{\lambda_{\nu(T)}}{e^{\alpha T}} + \alpha \int_0^T \frac{\lambda_{\nu(t)}}{e^{\alpha t}}\, dt. \tag{3.2}$$

If the left-hand side of (3.2) is hounded as T and consequently $N \to \infty$, the integral on the right does not exceed the values on the left and hence

$$\int_0^\infty \lambda_{\nu(t)}\, e^{-\alpha t}\, dt$$

converges. Suppose, ou the other hand, that

$$\int_0^\infty \lambda_{\nu(t)}\, e^{-\alpha t}\, dt$$

converges; then, since $\lambda_{\nu(t)}$ increases, we have, for $\beta > 0$,

$$\frac{(1-e^{\alpha\beta})\,\lambda_{\nu(T)}}{\alpha\, e^{\alpha T}} = \lambda_{\nu(T)} \int_T^{T+\beta} \frac{dt}{e^{\alpha t}} \leq \int_T^{T+\beta} \frac{\lambda_{\nu(t)}}{e^{\alpha t}}\, dt \leq \int_0^T \frac{\lambda_{\nu(t)}}{e^{\alpha t}}\, dt,$$

or, we find that

$$\frac{\lambda_{\nu(T)}}{e^{\alpha T}} = O(1),$$

and so the right-hand side of (3.2) is hounded and this gives the convergence of (3.1). Arguments for divergence can similarly be disposed of.

Lemma 3: *The series* (3.1) *and the series*:

$$\sum_{n=1}^{\infty} \exp\left[-\frac{\alpha}{\lambda_n}\left\{(\lambda_1-\lambda_0)\,\chi_1 + \cdots + (\lambda_n - \lambda_{n-1})\,\chi_n\right\}\right] \tag{3.3}$$

converge or diverge together.

For, let (3.1) he divergent. Then, since χ_n increases, we find that the series (3.3) is

$$> K + (m+\varepsilon)^{-1} \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1})\, e^{-\alpha\chi_n}, \quad (K = \text{a constant})$$

and so (3.3) diverges. Next suppose that (3.1) is convergent. Now

$$-\frac{\alpha}{\lambda_n}\sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})\,\chi_i = -\frac{\alpha}{\lambda_n}\left[\lambda_n\,\chi_n - \int_0^{\chi_n} \lambda_{\nu(x)}\, dx\right], \quad \chi_n = \sigma; \quad \lambda_{\nu(\sigma)} = \lambda_n$$

$$< -\alpha\,\chi_n + \alpha\,\Theta(\chi_n), \quad \Theta(x) = \frac{\log \mu(x)}{\lambda_{\nu(x)}}.$$

But as (3.1) is convergent, we have $\lambda_{\nu(T)} = O(e^{\alpha T})$, for all large T. Hence for large T

$$\frac{\log \mu(T)}{\lambda_{\nu(T)}} = o(1) + O\left(\frac{1}{\lambda_{\nu(T)}} \int_0^T e^{\alpha x}\, dx\right)$$

$$= O(1), \text{ for all large } T;$$

and that for all large T,

$$\frac{\log \mu(T)}{e^{\alpha T}} < K.$$

Consequently

$$\sum_{n=1}^{N} \exp\left[-\frac{\alpha}{\lambda_n}\sum_{i=1}^{n}(\lambda_i-\lambda_{i-1})\,\chi_i\right] < \sum_{n=1}^{N} e^{-\alpha\chi_n + \alpha\Theta(\chi_n)}$$

$$< K + (h-\varepsilon)\, K \sum_{n=1}^{N}(\lambda_n-\lambda_{n-1})\, e^{-\alpha\chi_n}$$

$$= K + K\left[e^{-\alpha T}\lambda_{\nu(T)} + \alpha\int_0^T \lambda_{\nu(x)}\, c^{-\alpha x}\, dx\right].$$

Therefore the series (3.3) is convergent.

Lemma 4 : *The integrals :*

$$\int_{\sigma_0}^{\infty} \frac{\log\mu(x)}{e^{\alpha x}}\, dx\, ; \quad \int_{\sigma_0}^{\infty} \frac{\lambda_{\nu(x)}\, dx}{e^{\alpha x}} \qquad (\mu(\sigma_0) \neq 0)$$

converge or diverge together.

This is easy to establish from the relation :

$$\text{(3.4)} \qquad \int_{\sigma_0}^{\sigma} \frac{\log\mu(x)}{e^{\alpha x}}\, dx - \frac{\log\mu(\sigma_0)}{\alpha\, e^{\alpha\sigma_0}} + \frac{\log\mu(\sigma)}{\alpha\, e^{\alpha\sigma}} = \frac{1}{\alpha}\int_{\sigma_0}^{\sigma} \frac{\lambda_{\nu(\nu)}}{e^{\alpha x}}\, dx,$$

which can be, in turn, obtained from

$$\log\mu(x) = \log\mu(x_0) + \int_{x_0}^{x} \lambda_{\nu(x)}\, dx_0\, .$$

Lemma 5 : *The integrals :*

$$\int_{\sigma_0}^{\infty} \frac{\log M(x)}{e^{\alpha x}}\, dx\, ; \qquad \int_{\sigma_0}^{\infty} \frac{\lambda_{\nu(x)}}{e^{\alpha x}}\, dx.$$

converge or diverge together.

For, the equation (3.4) can be written for large o, since $\log M(o) \sim \log\mu(a)$,

$$\int_{\sigma_0}^{\sigma} \frac{\log M(x)}{e^{\alpha x}}\, dx + \frac{\log M(\sigma)}{\alpha\, e^{\alpha\sigma}} = \Psi(\sigma)\left[K + \frac{1}{\alpha}\int_{\sigma_0}^{\sigma} \frac{\lambda_{\nu(x)}}{e^{\alpha x}}\, dx\right]$$

where $\Psi(o)$ is confined between two positive finite limits, and then the result follows exactly as in the preceding lemma.

Proof of the Main Theorem : From the definition of χ_n, it follows :

$$\exp\left[-\frac{\alpha}{\lambda_n}\left\{(\lambda_1-\lambda_0)\chi_1+\cdots+(\lambda_n-\lambda_{n-1})\chi_n\right\}\right]=\exp\left[-\frac{\alpha}{\lambda_n}\log\left|\frac{a_0}{a_n}\right|\right]=|a_n|^{\alpha/\lambda_n},$$

where we have, as we may, supposed that $|a_0|=1$. The theorem now follows hy combining the lemmas 2, 3 and 5.

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ÖZET

Bu araştırmada bir basamak fonksiyonuna tekabül ettirilen bir serinin yakınsaklık veya ıraksaklığını belirtmeğe yarıyan bir kriter, fonksiyonun karakteristik büyüklüklerini ihtiva eden bir integral yardımiyle ifade edilmiştir.