CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Some properties of convex functions are proved and the resulting formulae are used to establish some inequalities for such functions.

1. Let $M(u)$ be an increasing function of u. Suppose further that $\varphi(u)$ is an increasing function of u, absolutely continuous for $0 < u < \infty$ ($u = 0$ is an admissible value in some cases). Obviously $\varphi'(u)$ exists and is > 0. Let us suppose that $M(u)$ is a convex function with respect to $\varphi(u)$. Then if $u_1 < u < u_2$, we have:

$$M(u) \leq \frac{\varphi(u_2) - \varphi(u)}{\varphi(u_2) - \varphi(u_1)} M(u_1) + \frac{\varphi(u) - \varphi(u_1)}{\varphi(u_2) - \varphi(u_1)} M(u_2).$$

The preceding inequality can be written down as:

$$\text{(1)} \qquad \frac{M(u) - M(u_1)}{\varphi(u) - \varphi(u_1)} \leq \frac{M(u_2) - M(u_1)}{\varphi(u_2) - \varphi(u_1)} \leq \frac{M(u_2) - M(u)}{\varphi(u_2) - \varphi(u)}.$$

Let $0 < h_1 < h_2$. Then (1) yields, neglecting the obvious steps,

$$\text{(1')} \qquad \frac{M(u) - M(u - h_2)}{\varphi(u) - \varphi(u - h_2)} \leq \frac{M(u) - M(u - h_1)}{\varphi(u) - \varphi(u - h_1)} \leq \frac{M(u + h_1) - M(u)}{\varphi(u + h_1) - \varphi(u)}$$

$$\leq \frac{M(u + h_2) - M(u)}{\varphi(u + h_2) - \varphi(u)}.$$

If follows then that

$$\frac{M(u) - M(u - h)}{h} \left[\frac{\varphi(u) - \varphi(u - h)}{h}\right]^{-1}$$

does not decrease as $h \to +0$ and so has a limit $n_-(u)$. Similarly

$$\frac{M(u + h) - M(u)}{h} \left[\frac{\varphi(u + h) - \varphi(u)}{h}\right]^{-1}$$

does not increase as $h \to +0$ and so has a limit $n_+(u)$ and therefore

$$n_-(u) \leq n_+(u). \tag{2}$$

Next we show:

The right-hand derivative $n_+(u)$ of the continuous convex function $M(u)$ with respect to $\varphi(u)$ is a non-decreasing continuous function from the right.

Let $u_1 < u_2$. Then for arbitrarily small $h < 0$, $u_1 < u_1 + h < u_2 - h < u_2$, and so

$$\frac{M(u_1+h)-M(u_1)}{h}\left[\frac{\varphi(u_1+h)-\varphi(u_1)}{h}\right]^{-1} \leq \frac{M(u_2)-M(u_2-h)}{h}\left[\frac{\varphi(u_2)-\varphi(u_2-h)}{h}\right]^{-1}$$

and consequently we have, for $h \to 0$,

$$\begin{aligned} n_+(u_1) &\leq n_-(u_2) \\ &\leq n_+(u_2), \end{aligned} \tag{3}$$

from (2). Now we have, for $h > 0$, from (1′) that

$$n_+(u) \leq \frac{M(u+h)-M(u)}{h} \cdot \left[\frac{\varphi(u+h)-\varphi(u)}{h}\right]^{-1}.$$

Let us fix h and take the limit as $u \to u_0 + 0$; we have, since $M(u)$ and $\varphi(n)$ are continuous,

$$\lim_{u \to u_0+0} n_+(u) \leq \frac{M(u_0+h)-M(u_0)}{h}\left[\frac{\varphi(u_0+h)-\varphi(u_0)}{h}\right]^{-1}.$$

The limit on the left-hand side exists in view of the monotonocity of the function $n_+(u)$. Now let $h \to +0$, then

$$\lim_{u \to u_0+0} n_+(u) \leq n_+(n_0)$$

therefore

$$\lim_{u \to u_0+0} n_+(u) = n_+(u_0),$$

and so $n_+(u)$ is continuous from the right. Similarly, it can be shown that $n_-(u)$ is a non-decreasing continuous function from the left.

Next we shall show that $M(n)$ is an absolutely continuous function. Consider any interval $[a, b]$, $a > 0$; $b < \infty$. Let $a < u_1 < u_2 < b$. In view of (1′), we have

$$n_+(a) \leq \frac{M(u_2)-M(u_1)}{\varphi(u_2)-\varphi(u_1)} \leq n_-(b),$$

and since $\varphi(u)$ is absolutely continuous, we find that

$$\sum | M(u_2) - M(u_1) |$$

(where Σ denotes summation over $[a, b]$ for all finite pairs (u_1, u_2)), is arbitrarily small and it gives the absolute continuity. It can also be seen that if $\varphi(x)$ satisfies the LIPSCHITZ-condition ([1], p. 216), so does the function $M(u)$ in $[a, b]$.

Now suppose $M(a) = 0$, then

(4) $$M(u) = \int_a^u n(t)\, d\varphi(t).$$

To prove it, let us suppose $u_2 > u_1$, then from (2) and (3), we have:

(5) $$n_-(u_2) \geq n_+(u_1) \geq n_-(u_1).$$

But $n_-(u)$, being monotonic, is continuous almost everywhere. If u_1 is a point of continuity of $n_-(u)$, we have from (5), on letting $u_2 \to u_1$,

$$n_-(u_1) \geq n_+(u_1) \geq n_-(u_1),$$

and so $n_-(u_1) = n_+(u_1)$, and consequently

$$M'(u) = n(u)\, d\varphi(u),$$

almost everywhere. But since «every absolutely continuous function is an indefinite integral of its own derivative», ([1], p. 255), we find that the result (4) is established.

2. Some important applications of the result (4).

Here we apply the result (4) to find certain more general results whose particular cases are known for the case covering entire functions represented by TAYLOR's series and DIRICHLET-series. In fact, we shall, however, confine ourselves to the results regarding the growths of $M(u)$; $n(u)$ with respect to certain functions we will he defining. To obtain the results very precisely, let us introduce a function $\varrho(r)$ to satisfy the following conditions:

(i) $\varrho(r) \to \varrho$; $r \to \infty$; where $0 < \varrho < \infty$;

(ii) $$\frac{\varrho'(r)\, \varphi(r)}{\varphi'(r)} \to 0,$$

uniformly as $r \to \infty$;

(iii) $$\overline{\lim_{r\to\infty}} \frac{M(r)}{\exp\{\varrho(r)\,\varphi(r)\}} = 1.$$

Let

$$M_1(r) = \int_\delta^r \varrho(x)\, n(x)\, d\varphi(x); \qquad \delta > 0.$$

Then $M(r) \sim 1/\varrho\, M_1(r), \quad r \to \infty$. Define :

$$\overline{\lim_{r\to\infty}} \frac{M(r)}{g(r)} = \frac{A}{B}; \qquad \overline{\lim_{r\to\infty}} \frac{n(r)}{g(r)} = \frac{C}{D};$$

where

$$g(r) = \exp \int_\delta^r \varrho(x)\, d\varphi(x).$$

Theorem 1. *We shall prove :*

(i) $B \leq \frac{D}{\varrho}\left\{1 + \log\left(\frac{C}{D}\right)\right\}$; (ii) $A \leq \frac{C}{\varrho}$;

(iii) $A \geq \frac{C}{\varrho\, e} e^{D/C}$; (iv) $B \leq \frac{D}{\varrho}$.

Proof : To prove the results, let us suppose $R = R(r, K)$ ($K =$ a positive constant) such that $\varphi(R) - \varphi(r) \to \varphi(K) \geq 0$, which is always possible for a proper choice of $\varphi(r)$ and then of R [for example, $\varphi(r) = r$, $R = r + K$; $K > 0$ and $\varphi(r) = \log r$, $R = rK$; $K > 1$, etc.]. We shall use the following result:

(6) $$\frac{g(R)}{g(r)} \to e^{\varrho\varphi(K)},$$

uniformly as $r \to \infty$, and the proof of this is straight forward. Now

$$\begin{aligned}
M(R) &\sim \frac{1}{\varrho} M_1(R) \\
&= \frac{1}{\varrho}\int_\delta^r \varrho(x)\, n(x)\, d\varphi(x) + \frac{1}{\varrho}\int_r^R \varrho(x)\, n(x)\, d\varphi(x) \\
&= \frac{1}{\varrho}\int_\delta^r \frac{n(x)}{g(x)} g'(x)\, dx + \frac{1}{\varrho}\int_r^R n(x) \frac{g'(x)}{g(x)}\, dx \\
&< O(1) + \left(\frac{C+\varepsilon}{\varrho}\right) g(r) + \frac{1}{\varrho} n(R) \log\left\{\frac{g(R)}{g(r)}\right\} \\
&\sim \left(\frac{C+\varepsilon}{\varrho}\right) g(r) + n(R)\, \varphi(K).
\end{aligned}$$

Hence

(7) $$A \leq \frac{C}{\varrho} e^{-\varrho \varphi(K)} + C \varphi(K);$$

(8) $$B \leq \frac{C}{\varrho} e^{-\varrho \varphi(K)} + D \varphi(K).$$

The inequalities (7) and (8) will he the best possible in the sense that they are replaced (right hand expressions) by their minimum values respectively for some $\varphi(K)$. These values are respectively 0 and $1/\delta \log (C/D)$, and substituting these values of $\varphi(K)$ in (7) and (8) we get (*ii*) and (*i*) respectively. Further it can also be verified easily that for sufficiently large r,

$$M(R) > \left(\frac{D-\varepsilon}{\varrho}\right) g(r) + n(r)\, \varphi(K),$$

and so

(9) $$A \geq \frac{D}{\varrho} e^{-\varrho \varphi(K)} + C \varphi(K)\, e^{-\varrho \varphi(K)},$$

(10) $$B \geq \frac{D}{\varrho} e^{-\varrho \varphi(K)} + D \varphi(K)\, e^{-\varrho \varphi(K)}.$$

The maximum of the right-hand expression in (9) occurs at $\varphi(K) = (C-D)/\varrho C$, and of the right-hand expression in (10) at $\varphi(K) = 0$, and so (*iii*) and (*iv*) are proved.

Corollary. If $C = D$, then $A = B = C/\varrho$.

This easily follows from (*i*), (*ii*), (*iii*) and (*iv*).

We now show:

Theorem 2. *If* $A = B$, *then* $C = D = \varrho A$.

Proof: We have, for $r > r_0$ and $\varepsilon > 0$,

$$A - \varepsilon < \frac{M(r)}{g(r)} < A + \varepsilon.$$

Now

$$\Big(\varphi(R) - \varphi(r)\Big) n(r) = n(r) \int_r^R d\varphi(x) \leq \int_r^R n(x)\, d\varphi(x) = M(R) - M(r)$$

$$= \Big(1 + o(1)\Big) A\, g(R) - \Big(1 + o(1)\Big) A\, g(r)$$

$$= \Big(1 + o(1)\Big) (e^{\varrho \varphi(K)} - 1)\, A g(r)$$

$$= \Big(1 + o(1)\Big) \Big\{ \varrho\, \varphi(K) + O\Big((\varphi(K))^2\Big) \Big\} A\, g(r),$$

where $\varphi(K)$ is assumed to be small. Hence

$$\lim_{r\to\infty} \frac{n(r)}{g(r)} \leq \frac{\left\{\varrho\,\varphi(K) + O\left((\varphi(K))^2\right)\right\} A}{\varphi(K)},$$

and on making $\varphi(K) \to 0$ (on suitably choosing $\varphi(x)$ and K), we have:

$$\lim_{r\to\infty} \frac{n(r)}{g(r)} \leq \varrho A;$$

similarly it can be proved that

$$\lim_{r\to\infty} \frac{n(r)}{g(r)} \geq \varrho A,$$

and the result follows.

3. It is obvious from (4) that

$$M(R) < \int_r^R n(t)\,\varphi'(t)\,dt,$$

where $R = R(r) > r$ is chosen in such a way so as to satisfy

$$\varphi(R) - \varphi(r) \sim K\varphi(r) \qquad \text{or} \qquad \sim K + \varphi(r),$$

as $r \to \infty$, K is some positive constant (this always exists; for let $\varphi(r) = \log r$, $R = r^2$, $\varphi(r) = \log\log r$, $R = e^{(\log r)^2}$) and we find that

$$\frac{M(R)}{\varphi(R)} > K\,n(r) \to \infty \quad \text{as} \quad r \to \infty,$$

and so it is always that

$$\lim_{r\to\infty} \frac{M(r)}{\varphi(r)} = \infty.$$

But

$$\lim_{r\to\infty} \frac{\log M(r)}{\varphi(r)}$$

may or may not tend to ∞; in this article we consider only the latter possibility. Consequently define:

$$\text{(11)} \qquad \overline{\underline{\lim}}_{r\to\infty} \frac{\log M(r)}{\varphi(r)} = {\varrho \atop \lambda}; \quad 0 < \lambda, \quad \varrho < \infty.$$

We prove:

Theorem 3. *We shall have:*

$$\underline{\lim}_{r\to\infty} \frac{M(r)}{n(r)} \leq \frac{1}{\varrho} \leq \frac{1}{\lambda} \leq \overline{\lim}_{r\to\infty} \frac{M(r)}{n(r)}.$$

Proof: In the course of the proof we shall use the following result :

(12) $$\overline{\lim}_{r\to\infty} \frac{\log n(r)}{\varphi(r)} = \frac{\varrho}{\lambda};$$

and this can easily be proved by suitably choosing R such that $\varphi(R) - \varphi(r) \to \varphi(K)$ and making use of (4) and (11). First we show that

(13) $$\underline{\lim}_{r\to\infty} \frac{M(r)}{n(r)} \leq \frac{1}{\varrho}.$$

Suppose on the contrary that (13) is false, then given $A < \varrho$ and for all $r > r_0$

(14) $$n(r) \leq A\, M(r).$$

Hence

$$A \int_{r_0}^{r} e^{-\eta\varphi(x)} M(x)\, d\varphi(x) > O(1) + e^{-\eta\varphi(r)} M(r) + \eta \int_{r_0}^{r} e^{-\eta\varphi(x)} M(x)\, d\varphi(x)$$

where $A < \eta < \varrho$. Therefore

(15) $$(A - \eta) \int_{r_0}^{r} e^{-\eta\varphi(x)} M(x)\, d\varphi(x) > O(1) + \frac{1}{A} n(r)\, e^{-\eta\varphi(r)}.$$

But from (12) $n(r) > e^{\eta'\varphi(r)}$ for a sequence of arbitrarily large values of r and where $\eta < \eta' < \varrho$. But $(A - \eta)$ is a negative quantity and the integral in (15) is a positive quantity, since

$$e^{-\eta\varphi(x)} M(x)\, \varphi'(x).$$

Therefore (15) gives a contradiction ; hence (13) must be true. Similarly, we can show that

$$\overline{\lim}_{r\to\infty} M(r) / n(r) \geq \lambda^{-1},$$

and the result is proved.

REFERENCE

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ÖZET

Bu yazıda Konveks fonksiyonların bazı özelikleri incelenmekte ve bulunan formüller bu çeşit fonksiyonlar tarafından gerçeklenen birkaç eşitsizliğin elde edilmesinde kullanılmaktadır.