STEADY FLOWS OF TWO CONDUCTING INCOMPRESSIBLE AND IMMISCIBLE FLUIDS

J. B. SHUKLA AND R. PRASAD(*)

The flow of two conducting viscous, incompressible and immiscible fluids between two non-couducting parallel plates in the presence of a uniform transverse magnetic field has been discussed. For a large magnetic field, it has been shown that the flow fluxes for the two fluids, the interface velocity and interface induced magnetic field decrease as the strength of the magnetic field increases, but the skin frictions at the two plates do not depend upon this. When the strength of the magnetic field approaches zero, we get well known results for non-conducting fluids.

1. Introduction. The flow of two incompressible and immiscible fluids between two plates has been considered in [1]. Recently flow of n-immiscible fluids occupying different heights between two plates has been considered by KAPUR and SHUKLA [2]. They have shown that whatever be the number of fluids and whatever he their heights, a unique velocity maximum always exists, and a formula for finding the layer in which this maximum occurs has also been derived. The same authors [3] have also considered the flow of two conducting fluids between two plates in the presence of a transverse magnetic field, neglecting the induced electromagnetic effects. They have studied the effects of the magnetic field on the fluids flow and investigated the case of the maximum velocity. Here, we have considered the flow of two conducting, incompressible and immiscible fluids between two parallel nonconducting plates in the presence of a transverse magnetic field when the induced electromagnetic effects are taken into account. The main aim here is to discuss the effects of the applied magnetic field, viscosities and conductivities of the two fluids on the induced interface magnetic field.

2. Basic Equations. Consider the flow of two conducting, incompressible and immiscible fluids, between two non-coneucting, parallel stationary planes, separated by a distance $2h$ apart. A uniform magnetic field is applied in

(*) The authors are thankful to Prof. J. N. KAPUR for his guidance and advice in the preparation of this paper.

a transverse direction. Let μ_1, μ_2, σ_1, σ_2, μe_1, μe_2, denote the coefficients of viscosity, conductivity and permeability respectively of the two fluids each accupying a height h.

We take the x-axis along the interface and the y-axis perpendicular to it drawn into the first fluid. Let us suppose that a constant pressure gradient is applied to both the fluids. The physical situation is illustrated in the following figure

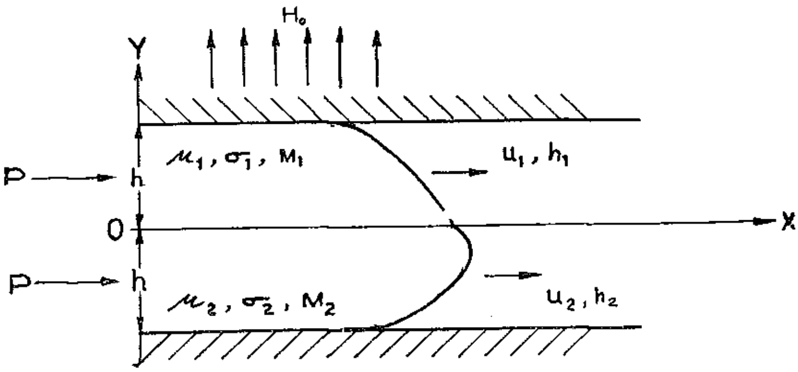

Fig. 1

Flow of two conducting immiscible fluids : Velocity profile.

The basic equations governing the flow of the conducting fluids are [4]

$$\frac{d^2u_i}{dy^2}+\frac{\mu e_i}{\mu_i}H_0\frac{dh_i}{dy}=-\frac{p}{\mu_i} \tag{1}$$

$$\frac{d^2h_i}{dy^2}+\sigma_i\,\mu e_i\,H_0\frac{du_i}{dy}=0 \tag{2}$$

$$i=1,2$$

where u_i and h_i are the velocities and the induced magnetic fields for the two fluids and

$$\frac{dp}{dx}=-P.$$

Since the tangential components of velocity and magnetic field should be continuous at the interface, the boundary conditions for u and h are as follows

$$\begin{cases} u_1=0\ , & h_1=0 \quad \text{at} \quad y=h \\ u_1=u_0\ , & h_1=h_0 \quad \text{at} \quad y=0 \end{cases} \tag{3}$$

$$\begin{cases} u_2=0\ , & h_2=0 \quad \text{at} \quad y=-h \\ u_2=u_0\ , & h_2=h_0 \quad \text{at} \quad y=0 \end{cases} \tag{4}$$

where u_0 and h_0 are the velocity and magnetic field at the interface.

Also, since the normal component of magnetic induction should he continuous, we have $\mu e_1 = \mu e_2 = \mu e$.

Solving equations (1) and (2) using the boundary conditions (3) we have the expressions for velocity and induced magnetic field for the first fluid as follows :

$$(5) \quad u_1 = \frac{1}{(2\cosh M_1 - 1)}\left[u_0\left\{\cosh M_1 - 1 - \cosh\frac{M_1}{h}y + \cosh M_1\left(1-\frac{y}{h}\right\}\right)\right.$$

$$\left. -\frac{1}{\sqrt{\sigma_1\mu_1}}\left(h_0 - \frac{Ph}{H_0\,\mu e}\right)\left\{\sinh M_1 - \sinh\frac{M_1 y}{h} - \sinh M_1\left(1-\frac{y}{h}\right)\right\}\right],$$

$$(6) \quad h_1 = \frac{1}{2(\cosh M_1 - 1)}\left[u_0\sqrt{\sigma_1\mu_1}\left\{\sinh\frac{M_1 y}{h} - \sinh M_1 + \sinh M_1\left(1-\frac{y}{h}\right)\right\}\right.$$

$$+\left(h_0 - \frac{Ph}{H_0\,\mu e}\right)\left\{1 - \cosh M_1 - \cosh\frac{M_1 y}{h} + \cosh M_1\left(1-\frac{y}{h}\right)\right\} + h_0 - \frac{P}{He\,\mu e}\,y.$$

Similarly from (1), (2) and (4) we have expressions for the second fluid as follows

$$(7) \quad u_2 = \frac{1}{2(\cosh M_2 - 1)}\left[u_0\left\{\cosh M_2 - \cosh\frac{M_2 y}{h} + \cosh M_2\left(1+\frac{y}{h}\right) - 1\right\}\right.$$

$$\left. +\frac{1}{\sqrt{\sigma_2\mu_2}}\left(h_0 + \frac{Ph}{H_0\,\mu e}\right)\left\{\sinh M_2 + \sinh\frac{M_2 y}{h} - \sinh M_2\left(1+\frac{y}{h}\right)\right\}\right],$$

$$(8) \quad h_2 = \frac{1}{2(\cosh M_2 - 1)}\left[u_0\sqrt{\sigma_2\mu_2}\left\{\sinh M_2 + \sinh\frac{M_2 y}{h} - \sinh M_2\left(1+\frac{y}{h}\right)\right\}\right.$$

$$\left. +\left(h_0 + \frac{Ph}{H_0\,\mu e}\right)\left\{1 - \cosh M_2 - \cosh\frac{M_2 y}{h} + \cosh M_2\left(1+\frac{y}{h}\right)\right\}\right] + h_0 - \frac{P}{H_0\,\mu e}\,y$$

where M_1 and M_2 are the HARTMANN numbers for the two flows.

By using the following transformations

$$(9) \quad \left\{\begin{array}{lll} \bar{u}_1 = \dfrac{u_1\,\mu_1}{Ph^2} & , & \bar{h}_1 = \dfrac{h_1}{Ph^2}\sqrt{\dfrac{\mu_1}{\sigma_1}}, \\[2ex] \bar{u}_2 = \dfrac{u_2\,\mu_2}{Ph^2} & , & \bar{h}_2 = \dfrac{h_2}{Ph^2}\sqrt{\dfrac{\mu_2}{\sigma_2}}, \\[2ex] \bar{u}_0 = \dfrac{u_0\,\mu_1}{Ph_2} & , & \bar{h}_0 = \dfrac{h_0}{Ph^2}\sqrt{\dfrac{\mu_1}{\sigma_1}}, \\[2ex] \lambda^2 = \dfrac{\mu_2}{\mu_1} & & \\[2ex] & , & \dfrac{\sigma_2}{\sigma_1} = \mu^2 = \dfrac{\lambda_2 M_2^2}{M_1^2}\,. \\[2ex] \bar{y} = \dfrac{y}{h} & & \end{array}\right.$$

the equations (5), (6), (7) and (8) are reduced to non dimensional forms as follows

$$\bar{u}_1 = \frac{1}{2(\cosh M_1 - 1)} \Big[\bar{u}_0 \Big\{ \cosh M_1 - 1 - \cosh M_1 \bar{y} + \cosh M_1 (1 - \bar{y}) \Big\} - \Big(\bar{h}_0 - \frac{1}{M_1} \Big) \Big\{ \sinh M_1 - \sinh M_1 \bar{y} - \sinh M_1 (1 - \bar{y}) \Big\} \Big], \tag{10}$$

$$\bar{h}_1 = \frac{1}{2(\cosh M_1 - 1)} \Big[\bar{u}_0 \Big\{ \sinh M_1 \bar{y} - \sinh M_1 + \sinh M_1 (1 - \bar{y}) \Big\} + \Big(\bar{h}_0 - \frac{1}{M_1} \Big) \Big\{ 1 - \cosh M_1 - \cosh M_1 \bar{y} + \cosh M_1 (1 - \bar{y}) \Big\} \Big] + \bar{h}_0 - \frac{1}{M_1} \bar{y}. \tag{11}$$

$$\bar{u}_2 = \frac{1}{2(\cosh M_2 - 1)} \Big[\lambda^2 \bar{u}_0 \Big\{ \cosh M_2 - 1 - \cosh M_2 \bar{y} + \cosh M_2 (1 + \bar{y}) \Big\} + \Big(\frac{M_1}{M_2} \bar{h}_0 + \frac{1}{M_2} \Big) \Big\{ \sinh M_2 + \sinh M_2 \bar{y} - \sinh M_2 (1 + \bar{y}) \Big\} \Big] \tag{12}$$

and

$$\bar{h}_2 = \frac{1}{2(\cosh M_2 - 1)} \Big[\lambda^2 \bar{u}_0 \Big\{ \sinh M_2 + \sinh M_2 \bar{y} - \sinh M_2 (1 + \bar{y}) \Big\} + \Big(\frac{M_1}{M_2} \bar{h}_0 + \frac{1}{M_2} \Big) \Big\{ 1 - \cosh M_2 - \cosh M_2 \bar{y} + \cosh M_2 (1 + \bar{y}) \Big\} \Big] + \frac{M_1}{M_2} \bar{h}_0 - \frac{1}{M_2} \bar{y}. \tag{13}$$

3. Determination of interface velocity and magnetic field. For determining interface velocity u_0 we assume that the shearing stresses are continuous at the interface; the condition for this is

$$\mu_1 \left(\frac{du_1}{dy} \right)_{y=0} = \mu_2 \left(\frac{du_2}{dy} \right)_{y=0}$$

which can be written in the dimensionless form as

$$\left(\frac{d\bar{u}_1}{d\bar{y}} \right)_{\bar{y}=0} = \left(\frac{d\bar{u}_2}{d\bar{y}} \right)_{\bar{y}=0}. \tag{14}$$

Then from (10), (12) and (14) we have,

$$\bar{u}_0 = \frac{1}{\dfrac{\frac{M_1}{2}}{\tanh \frac{M_1}{2}} + \lambda^2 \dfrac{\frac{M_2}{2}}{\tanh \frac{M_2}{2}}}. \tag{15}$$

Now for determining h_0 we use the continuity of the tangential component of the electric field, the condition for which in non dimensional form is

(16) $$\lambda^2 \left(\frac{d\bar{h}_1}{d\bar{y}}\right)_{\bar{y}=0} = \frac{M_1}{M_2}\left(\frac{d\bar{h}_2}{d\bar{y}}\right)_{\bar{y}=0} .$$

Then again from (11), (13) and (16) we get the interface magnetic field as follows

(17) $$\bar{h}_0 = \frac{1}{M_1}\,\frac{\left(\dfrac{\frac{M_1}{2}}{\tanh\frac{M_1}{2}} - 1\right) - \left(\dfrac{M_1}{\lambda M_2}\right)^2\left(\dfrac{\frac{M_2}{2}}{\tanh\frac{M_2}{2}} - 1\right)}{\dfrac{\frac{M_1}{2}}{\tanh\frac{M_1}{2}} + \left(\dfrac{M_1}{\lambda M_2}\right)^2 \dfrac{\frac{M_2}{2}}{\tanh\frac{M_2}{2}}} .$$

4. Skin frictions at the plates and flow fluxes. The skin friction at the upper plate is given by

$$\bar{\tau}_1 = \frac{\tau_1}{Ph} = -\left(\frac{d\bar{u}_1}{d\bar{y}}\right)_{\bar{y}=1}$$

which gives on using (10)

(18) $$\bar{\tau}_1 = -\frac{1}{2}(M_1\bar{h}_0 - 1) + \bar{u}_0 \frac{\frac{M_1}{2}}{\tanh\frac{M_1}{2}} .$$

Similarly the skin friction at the lower plate ($\bar{y} = -1$) is given by

(19) $$\bar{\tau}_2 = \frac{1}{2}(M_1\bar{h}_0 + 1) + \lambda^2 \bar{u}_0 \frac{\frac{M_2}{2}}{\tanh\frac{M_2}{2}} .$$

The flux for the first fluid is given by

(20) $$\bar{Q}_1 = \frac{Q_1 \mu_1}{Ph^3} = \int_0^1 \bar{u}_1\, d\bar{y}.$$

Then from (10) and (20) we get finally

(21) $$\bar{Q}_1 = \frac{\bar{u}_0}{2} - \frac{(M_1\bar{h}_0 - 1)}{M_1^2}\left(\frac{\frac{M_1}{2}}{\tanh\frac{M_1}{2}} - 1\right).$$

Similarly the flow flux for the second fluid is given by

(22) $$\bar{Q}_2 = \frac{Q_2 \mu_2}{Ph^3} = \frac{\lambda^2 \bar{u}_0}{2} + \frac{(M_1\bar{h}_0 + 1)}{M_2^2}\left(\frac{\frac{M_2}{2}}{\tanh\frac{M_2}{2}} - 1\right).$$

5. Results. Since $M/\tanh M$ increases as M increases, from equations (15) we see that the interface velocity decreases as M_1 and M_2 increases. But M_1 and M_2 can increase by increasing either, the strength of the magnetic field, or by increasing conductivities of the fluid Thus we infer that the interface velocity decreases as the strength of the magnetic field increases or as the conductivities of the two fluids increase.

To see the effect of viscosities, equation (15) can be written as

$$\frac{u_0}{Ph^2} = \frac{1}{\frac{k_1}{2}\sqrt{\mu_1}\coth\frac{k_1}{2\sqrt{\mu_1}} + \frac{k_2}{2}\sqrt{\mu_2}\coth\frac{k_2}{2\sqrt{\mu_2}}} \tag{23}$$

where k_1 and k_2 are independent of μ_1 and μ_2.

Since

$$\sqrt{\mu_1}\coth\frac{k_1}{2\sqrt{\mu_2}} \qquad \text{and} \qquad \sqrt{\mu_2}\coth\frac{k_2}{2\sqrt{\mu_2}}$$

respectively increase as μ_1 and μ_2 increase, we infer from (15) that the interface velocity decreases as viscosity of the two fluids increase.

For large M_1 and M_2 *i. e.* when the strength of the magnetic field is large, we get from equations (15), (17), (18), (19), (21) and (20),

$$\left\{\begin{array}{ll} \bar{u}_1 \approx \dfrac{1}{M_1}\dfrac{2}{1+\lambda\mu}, & \bar{h}_0 \approx \dfrac{1}{M_1}\dfrac{\lambda\mu-1}{\lambda\mu+1} \\ \bar{\tau}_1 \approx \dfrac{2}{1+\lambda\mu}, & \bar{\tau}_2 \approx \dfrac{2\lambda\mu}{1+\lambda\mu} \\ \bar{Q}_1 \approx \dfrac{1}{M_1}\dfrac{2}{1+\lambda\mu}, & \bar{Q}_2 \approx \dfrac{1}{M_1}\dfrac{2\lambda^2}{\lambda\mu+1} \end{array}\right. \tag{24}$$

From these equations it is obvious that the interface velocity, induced interface magnetic field and the flow fluxes of the two fluids, decrease as the strength of the magnetic field increase. But the skin frictions at the two plates are almost independent of the applied field. The induced magnetic field decreases because of the reduction of the interface velocity by the applied magnetic field. These results are in conformity with the stabilizing property of the magnetic field.

To study the effects of viscosities and conductivities, h_0 can be approximated for large magnetic field, as

$$h_0 \approx \frac{Ph}{H_0\mu e}\,\frac{1-\sqrt{\dfrac{\sigma_1\mu_1}{\sigma_2\mu_2}}}{1+\sqrt{\dfrac{\sigma_1\mu_1}{\sigma_2\mu_2}}} \tag{25}$$

From this it is clear that h_0 vanishes when $\sigma_1 \mu_1 = \sigma_2 \mu_2$. It increases as σ_2 or μ_2 increases and decreases as σ_1 or μ_1 increases.

Similarly the expressions for τ_1 and τ_2 can be approximated as

$$(26) \qquad \begin{cases} \tau_1 \approx \dfrac{2Ph}{\mu_1 + \sqrt{\dfrac{\mu_1 \mu_2 \sigma_2}{\sigma_1}}} \\ \tau_2 \approx \dfrac{2Ph}{\mu_2 + \sqrt{\dfrac{\sigma_1 \mu_1 \mu_2}{\sigma_2}}} \end{cases}$$

Here we can see that τ_1 decreases as σ_2, μ_2 or μ_1 increases for fixed σ_1 and it increases as σ_1 increases for fixed σ_2, μ_1, and μ_2. Similarly τ_2 decreases as σ_1, μ_1 or μ_2 increases for fixed σ_2 and it increases as σ_2 increases for fixed σ_1, μ_1, and μ_2. For the flow fluxes, we can similarly see, that Q_1 and Q_2 decreases as any of the quantities σ_1, σ_2, μ_1 or μ_2 increases.

When M_1 and M_2 are small we have,

$$(27) \qquad \begin{cases} \bar{u}_0 \approx \dfrac{1}{1+\lambda^2} \left[1 - \dfrac{M_1^2}{12} \dfrac{1+\mu^2}{1+\lambda^2}\right] \\ \bar{h}_0 \approx \dfrac{\mu^2}{\lambda^2} \dfrac{\lambda^2 - 1}{\mu^2 + 1} \dfrac{M_1}{12} \end{cases}$$

$$(28) \qquad \begin{cases} \bar{\tau}_1 \approx \dfrac{3}{2} - \dfrac{\lambda^2}{1+\lambda^2} - \dfrac{M_1^2}{12} \left\{ \dfrac{\mu^2 (\lambda^2 - 1)}{2\lambda^2(\mu^2+1)} + \dfrac{\mu^2 - \lambda^2}{(1+\lambda^2)^2} \right\} \\ \bar{\tau}_2 \approx \dfrac{1}{2} + \dfrac{\lambda^2}{1+\lambda^2} + \dfrac{M_1^2}{12} \left\{ \dfrac{\mu^2 (\lambda^2 - 1)}{2\lambda^2(\mu^2+1)} + \dfrac{\mu^2 - \lambda^2}{(1+\lambda^2)^2} \right\} \end{cases}$$

$$(29) \qquad \begin{cases} \bar{Q}_1 \approx \dfrac{1}{2(1+\lambda^2)} + \dfrac{1}{12} - \dfrac{M_1^2}{24} \left\{ \dfrac{1+\mu^2}{(1+\lambda^2)^2} + \dfrac{\mu^2}{6\lambda^2} \left(\dfrac{\lambda^2 - 1}{\mu^2 + 1} \right) \right\} \\ \bar{Q}_2 \approx \dfrac{\lambda^2}{2(1+\lambda^2)} + \dfrac{1}{12} - \dfrac{M_1^2}{24} \left\{ \dfrac{\lambda^2 (1+\mu^2)}{(1+\lambda^2)^2} - \dfrac{\mu^2}{6\lambda^2} \left(\dfrac{\lambda^2 - 1}{\mu^2 + 1} \right) \right\} \end{cases}$$

From these equations we can see that when $M_1 \to 0$ we get the results discrihed in [1].

REFERENCE

[1] BIRD, R. B.; STEWART, W. E. AND LIGHTFOOL, E. N. : **Transport phenomena**, JOHN WILEY AND SONS, 54 (1960).

[2] KAPUR, J. N. AND SHUKLA J. B. : *The flow of incompressible immiscible fluids between two plates.*

[3] KAPUR, J. N. AND SHUKLA J. B. : *The flow of two conducting immiscible fluids between two parallel plates.*

Forthcoming in Revue de la Faculté des Sciences de l'Université d'Istanbul.

[4] PAI, S. I. : **Magnetodynamics and Plasma dynamics**, SPRINGER VERLAG (1962).

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR AND V.S.S.D. COLLEGE, KANPUR.

(Manuscript received March, 2, 1964)

ÖZET

Luzuciyeti haiz, iletken, sıkıştırılamayan ve karışamayan iki sıvının, iletken olmayan iki paralel levha arasındaki akımı, levhalara dik düzgün bir magnetik alanın varlığı halinde incelenmiştir. Büyük bir magnetik alan halinde, akıma dik magnetik alanın şiddeti arttıkça, iki sıvının akım hızlarının, arayüzey hızının ve arayüzey üzerinde indüksiyondan doiayı meydana gelen magnetik alanın, azaldıkları görülmektedir. Buna mukabil iki levha üzerindeki yüzey sürtünmeleri bu alana tâbi değildir. Magnetik alanın şiddetinin sıfıra yaklaşması halinde, iletken olmayan sıvılar için eskiden beri bilinen sonuçlar tekrar elde edilmektedir.