ON AXIALLY SYMMETRIC SUPERPOSABLE FLOWS OF THE TYPE CURL $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, CURL $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$

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In the present paper we have discussed inviscid axially symmetric superposable flows of the type curl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, curl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$, having both poloidal and toroidal components of velocity field, when λ_1, λ_2 are functions of r and t and z and t respectively. We have also discussed steady viscous flows of the same type when λ_1 and λ_2 are functions of r and z respectively. Some special inviscid flows of the given type have also been discussed.

1. Introduction. Axially symmetric superposable flows have been studied by PREM PRAKASH [1], RAM BALLABH [2], BHATNAGAR and VARMA [3] and KAPUR [4,5], while flows of the type curl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, curl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$ which are mutually superposable have been studied by RAM BALLABH[6], GHILDYAL[7] and DEVI SINGH[8]. In the present paper, we study axially symmetric superposable flows of the above type when the flows have both poloidal and toroidal components.

In section 2 we first establish a theorem giving the condition for both the flows to be self-superposable when both λ_1, λ_2 are functions of one space variable and time. In section 3 we have shown that for inviscid flows of the given type λ_1 and λ_2 cannot be functions of both r and t. In section 4 we have obtained the most general flows of the given type when λ_1, λ_2 are functions of z and t alone. Sections 5 and 6 are devoted to a study of steady flows of the given type when λ_1, λ_2 are respectively functions of r and z alone. Finally in section 7, we consider some special inviscid flows of the given type.

2. A theorem on superposable flows of the type curl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, curl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$.

Theorem. *Let α_1, α_2, α_3 represent a curvilinear orthogonal system of co-ordinates. For superposable flows of the type carl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, carl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$ of incompressible viscous fluids, let λ_1, λ_2 both be functions of one of the three co-*

ordinates α_1, α_2, α_3 *and t alone; then each of the two motions whose velocity vectors are given by* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *is selfsuperposable if*

$$(\mathbf{q}_2 \cdot \nabla)\,\mathbf{q}_1 = (\mathbf{q}_1 \cdot \nabla)\,\mathbf{q}_2\,. \tag{1}$$

Proof:

$$\left\{\begin{aligned} \text{Curl}\;[\mathbf{q}_1 \times \text{curl}\,\mathbf{q}_1] &= \text{curl}\,[\mathbf{q}_1 \times \lambda_2\,\mathbf{q}_2] = \text{curl}\;[\lambda_2\,\mathbf{q}_1 \times \mathbf{q}_2] \\ &= \text{grad}\;\lambda_2 \times (\mathbf{q}_1 \times \mathbf{q}_2) + \lambda_2\,\text{curl}\,(\mathbf{q}_1 \times \mathbf{q}_2) \\ &= (\text{grad}\;\lambda_2 \cdot \mathbf{q}_2)\,\mathbf{q}_1 - (\text{grad}\,\lambda_2 \cdot \mathbf{q}_1)\,\mathbf{q}_2 \\ &\quad + \lambda_2\,[\mathbf{q}_1\,\text{div.}\,\mathbf{q}_2 - \mathbf{q}_2\,\text{div.}\,\mathbf{q}_1 + (\mathbf{q}_2 \cdot \nabla)\,\mathbf{q}_1 - (\mathbf{q}_1 \cdot \nabla)\,\mathbf{q}_2]. \end{aligned}\right. \tag{2}$$

Since the fluids are incompressible, we get from the equation of continuity

$$\text{div.}\;\mathbf{q}_1 = 0. \qquad \text{div.}\;\mathbf{q}_2 = 0. \tag{3}$$

Also

$$0 = \text{div.}\;(\text{curl}\;\mathbf{q}_1) = \text{div.}\;(\lambda_2\,\mathbf{q}_2) = \lambda_2\,\text{div.}\;\mathbf{q}_2 + \mathbf{q}_2 \cdot \text{grad}\,\lambda_2\,. \tag{4}$$

Using (3) we get

$$\mathbf{q}_2 \cdot \text{grad}\;\lambda_2 = 0, \tag{5}$$

and

$$\mathbf{q}_1 \cdot \text{grad}\;\lambda_1 = 0. \tag{6}$$

Using (3) and (5) we get from (2)

$$\text{curl}\;[\mathbf{q}_1 \times \text{curl}\;\mathbf{q}_1] = -\,(\text{grad}\;\lambda_2 \cdot \mathbf{q}_1)\,\mathbf{q}_2 + \lambda_2\,[(\mathbf{q}_2 \cdot \nabla)\,\mathbf{q}_1 - (\mathbf{q}_1 \cdot \nabla)\,\mathbf{q}_2]. \tag{7}$$

Similarly

$$\text{curl}\;[\mathbf{q}_2 \times \text{curl}\,\mathbf{q}_2] = -\,(\text{grad}\;\lambda \cdot \mathbf{q}_2)\,\mathbf{q}_1 + \lambda_1\,[(\mathbf{q}_1 \cdot \nabla)\,\mathbf{q}_2 - (\mathbf{q}_2 \cdot \nabla)\,\mathbf{q}_1]. \tag{8}$$

Now let λ_1, λ_2 be functions of α_1 and t alone and in the curvilinear orthogonal system let

$$ds^2 = h_1^2\,d\alpha_1^2 + h_2^2\,d\alpha_2^2 + h_3^2\,d\alpha_3^2 \tag{9}$$

where h_1, h_2 and h_3 are functions of α_1, α_2, α_3; then

$$\text{grad}\;\lambda_1 = \left(\frac{1}{h_1}\frac{\partial \lambda_1}{\partial \alpha_1},\;\; 0,\;\; 0\right), \quad \text{grad}\;\lambda_2 = \left(\frac{1}{h_1}\frac{\partial \lambda_2}{\partial \alpha_1},\;\; 0,\;\; 0\right). \tag{10}$$

From (5), (6), the components of $\mathbf{q}_1$, $\mathbf{q}_2$ in the curvilinear orthogonal system would be of the type

(11) $$\mathbf{q}_1 = (0,\ v_1,\ w_1), \qquad \mathbf{q}_2 = (0,\ v_2,\ w_2).$$

From (1), (10), (11), we have

(12) $$(\mathbf{q}_2 \cdot \nabla)\, \mathbf{q}_1 - (\mathbf{q}_1 \cdot \nabla)\, \mathbf{q}_2 = 0, \quad \operatorname{grad} \lambda_1 \cdot \mathbf{q}_2 = 0, \quad \operatorname{grad} \lambda_2 \cdot \mathbf{q}_1 = 0,$$

so that from (7) and (8)

(13) $$\operatorname{curl}\, [\mathbf{q}_1 \times \operatorname{curl}\, \mathbf{q}_1] = 0,$$

(14) $$\operatorname{curl}\, [\mathbf{q}_2 \times \operatorname{curl}\, \mathbf{q}_2] = 0.$$

Thus the two individual motions are self-superposable. The same result obviously holds when λ_1, λ_2 are functions of α_2, t alone or of α_3, t alone.

3. Axially-symmetric flows of the given type when λ_1, λ_2 are functions of r, t alone.

Let

(15) $$\mathbf{q}_i = -\frac{1}{r}\frac{\partial \psi_i}{\partial z}\, \mathbf{i}_r + \frac{\Omega_i}{r}\, \mathbf{i}_\vartheta + \frac{1}{r}\frac{\partial \psi_i}{\partial r}\, \mathbf{i}_z \qquad (i = 1,\ 2)$$

where $\mathbf{i}_r$, $\mathbf{i}_\vartheta$ $\mathbf{i}_z$ denote the unit vectors in the cylindrical system of co-ordinates. From (15)

(16) $$\operatorname{curl}\, \mathbf{q}_i = -\frac{1}{r}\frac{\partial \Omega_i}{\partial z}\, \mathbf{i}_r - \frac{1}{r} D^2 \psi_i\, \mathbf{i}_\vartheta + \frac{1}{r}\frac{\partial \Omega_i}{\partial r}\, \mathbf{i}_z, \qquad (i = 1,\ 2)$$

where

(17) $$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

From (5) and (6)

$$\frac{\partial \lambda_i}{\partial r}\left(-\frac{1}{r}\frac{\partial \psi_i}{\partial z}\right) = 0.$$

Since $\frac{\partial \lambda_i}{\partial r} \neq 0$, this implies

(18) $$\frac{\partial \psi_1}{\partial z} = 0, \qquad \frac{\partial \psi_2}{\partial z} = 0,$$

so that both the motions do note have radial components of velocity. From the relations

(19) $$\operatorname{curl}\, \mathbf{q}_1 = \lambda_2\, \mathbf{q}_2, \qquad \operatorname{curl}\, \mathbf{q}_2 = \lambda_1\, \mathbf{q}_1$$

we get, on using (15), (16), (17) and (18),

(20) $$\frac{\partial \Omega_1}{\partial z} = 0, \qquad \frac{\partial \Omega_2}{\partial z} = 0,$$

(21) $$D^2 \psi_1 = \frac{\partial^2 \psi_1}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_1}{\partial r} = -\lambda_2 \Omega_2, \qquad \frac{\partial \Omega_1}{\partial r} = \lambda_2 \frac{\partial \psi_2}{\partial r},$$

(22) $$D^2 \psi_2 = \frac{\partial^2 \psi_2}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_2}{\partial r} = -\lambda_1 \Omega_1, \qquad \frac{\partial \Omega_2}{\partial r} = \lambda_1 \frac{\partial \psi_1}{\partial r}.$$

Again in this case, on using (15), (18), (20) and axial symmetry, we get

(23) $$\left\{\begin{aligned} &(\mathbf{q}_2 \cdot \nabla)\,\mathbf{q}_1 - (\mathbf{q}_1 \cdot \nabla)\,\mathbf{q}_2 \\ &= \left[-\frac{1}{r}\frac{\partial \psi_2}{\partial z}\frac{\partial}{\partial r} + \frac{\Omega_2}{r^2}\frac{\partial}{\partial \vartheta} + \frac{1}{r}\frac{\partial \psi_2}{\partial r}\frac{\partial}{\partial z}\right]\left[-\frac{1}{r}\frac{\partial \psi_1}{\partial z}\mathbf{i}_r + \frac{\Omega_1}{r}\mathbf{i}_\vartheta + \frac{1}{r}\frac{\partial \psi_1}{\partial r}\mathbf{i}_z\right] \\ &- \left[-\frac{1}{r}\frac{\partial \psi_1}{\partial z}\frac{\partial}{\partial r} + \frac{\Omega_1}{r^2}\frac{\partial}{\partial \vartheta} + \frac{1}{r}\frac{\partial \psi_1}{\partial r}\frac{\partial}{\partial z}\right]\left[-\frac{1}{r}\frac{\partial \psi_2}{\partial z}\mathbf{i}_r + \frac{\Omega_2}{r}\mathbf{i}_\vartheta + \frac{1}{r}\frac{\partial \psi_2}{\partial r}\mathbf{i}_z\right] = 0 \end{aligned}\right.$$

so that the condition of the above theorem and equations (13) and (14) are satisfied.

Now the conditions of integrability for the two motions are

(24) $$\frac{\partial}{\partial t}(\operatorname{curl}\, \mathbf{q}_i) + \operatorname{curl}\,(\operatorname{curl}\, \mathbf{q}_i \times \mathbf{q}_i) = \nu \nabla^2 (\operatorname{curl}\, \mathbf{q}_i).$$

For inviscid liquids on using (13) and (14), these reduce to

(25) $$\frac{\partial}{\partial t}(\operatorname{curl}\, \mathbf{q}_i) = 0,$$

or

(26) $$\frac{\partial}{\partial t}\left(\lambda_i \frac{\Omega_i}{r}\right) = 0,$$

and

(27) $$\frac{\partial}{\partial t}\left(\frac{\lambda_i}{r}\frac{\partial \psi_i}{\partial r}\right) = 0.$$

From (21), (22) and (27)

(28) $$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial \Omega_i}{\partial r}\right) = 0.$$

Integrating (28), we get

(29) $$\Omega_i = f_i(r) + \varphi_i(t) \qquad (i = 1,\ 2)$$

where f_i, φ_i are arbitrary differentiable functions of r and t respectively. From (26), (29)

$$\lambda_i = \frac{P_i(r)}{f_i(r) + \varphi_i(t)} \qquad (i = 1,\ 2) \tag{30}$$

where P_i is again an arbitrary differentiable function of r. This equation gives the most general form which λ_1, λ_2 can have. From (21), (22), (29) and (30)

$$\frac{\partial \psi_1}{\partial r} = \frac{f_1(r) + \varphi_1(t)}{P_1(r)} f_2'(r)\,; \qquad \frac{\partial \psi_2}{\partial r} = \frac{f_2(r) + \varphi_2(t)}{P_2(r)} f_1'(r) \tag{31}$$

so that the general forms of velocity vectors are

$$\mathbf{q}_1 = \frac{1}{r}\left(f_1(r) + \varphi_1(t)\right) \mathbf{i}_\vartheta + \frac{1}{r}\,\frac{f_2'(r)}{P_1(r)}\left(f_1(r) + \varphi_1(t)\right) \mathbf{i}_z \tag{32}$$

$$\mathbf{q}_2 = \frac{1}{r}\left(f_2(r) + \varphi_2(t)\right) \mathbf{i}_\vartheta + \frac{1}{r}\,\frac{f_1'(r)}{P_2(r)}\left(f_2(r) + \varphi_2(t)\right) \mathbf{i}_z . \tag{33}$$

Equations (30), (32) and (33) give the general solution of our problem. The functions f_i, φ_i and P_i are however to be so chosen that ψ_i, Ω_i satisfy the first of the equations (21) and (22). From the first of these we get

$$f_2' P_1 - r P_1 f_2'' + r P_1' f_2' = 0, \tag{34}$$

$$r P_1^2 P_2 - f_1 f_2' P_1 + r P_1 (f_1 f_2'' + f_1' f_2') - r P_1' (f_1 f_2') = 0. \tag{35}$$

These equations give

$$-\frac{P_1'}{P_1} + \frac{f_2''}{f_2'} = \frac{1}{r}$$

and

$$P_1 P_2 = - f_1' f_2',$$

so that

$$\frac{f_2'}{P_1} = Ar, \qquad \frac{f_1'}{P_2} = -\frac{1}{Ar}. \tag{36}$$

Similarly from (22)

$$\frac{f_1'}{P_2} = Br, \qquad \frac{f_2'}{P_1} = -\frac{1}{Br}, \tag{37}$$

where A and B are arbitrary constants.

Equations (36) and (37) are not consistent and therefore show that no axially symmetric superposable motions of the given type are possible when λ_1, λ_2 are functions of r and t only. However steady motions of the given type exist when λ_1, λ_2 are functions of r only for then instead of four equations of the type (34) to (37), we shall have two equations to be satisfied by f_1, f_2, P_1 and P_2. We study these motions in the next section for the more general case of viscous fluids. For inviscid flows (21), (22) give four equations to solve for any four of the functions $\lambda_1, \lambda_2, \psi_1, \psi_2, \Omega_1, \Omega_2$ when any two of these are given. Some particular solutions for the inviscid case will he studied in the last section.

4. Axially-symmetric flows of the given type when λ_1, λ_2 are functions of z, t alone.

From (5), (6) and (15)

$$\frac{\partial \lambda_i}{\partial z}\left(\frac{1}{r}\frac{\partial \psi_i}{\partial r}\right)=0. \tag{38}$$

Since $\frac{\partial \lambda_i}{\partial z} \neq 0$, it means that

$$\frac{\partial \psi_1}{\partial r}=0, \qquad \frac{\partial \psi_2}{\partial r}=0, \tag{39}$$

so that hoth the motions do not have axial components of velocity. From the relations (19), on using (15), 16), (17) and (39), we get

$$\frac{\partial \Omega_1}{\partial r}=0, \qquad \frac{\partial \Omega_2}{\partial r}=0, \tag{40}$$

$$\frac{\partial^2 \psi_1}{\partial z^2}=-\lambda_2 \Omega_2 ; \qquad \lambda_2 \frac{\partial \psi_2}{\partial z}=\frac{\partial \Omega_1}{\partial z}, \tag{41}$$

$$\frac{\partial^2 \psi_2}{\partial z^2}=-\lambda_1 \Omega_1 ; \qquad \lambda_1 \frac{\partial \psi_1}{\partial z}=\frac{\partial \Omega_2}{\partial z}. \tag{42}$$

Again on using (15), (39), (40) and axial symmetry

$$\left\{\begin{aligned} &(\mathbf{q}_2\cdot\nabla)\,\mathbf{q}_1-(\mathbf{q}_1\cdot\nabla)\,\mathbf{q}_2\\ &=\left[-\frac{1}{r}\frac{\partial \psi_2}{\partial z}\frac{\partial}{\partial r}+\frac{\Omega_2}{r^2}\frac{\partial}{\partial \vartheta}+\frac{1}{r}\frac{\partial \psi_2}{\partial r}\frac{\partial}{\partial z}\right]\left[-\frac{1}{r}\frac{\partial \psi_1}{\partial z}\,\mathbf{i}_r+\frac{\Omega_1}{r}\,\mathbf{i}_\vartheta+\frac{1}{r}\frac{\partial \psi_1}{\partial r}\,\mathbf{i}_z\right]\\ &\quad-\left[-\frac{1}{r}\frac{\partial \psi_1}{\partial z}\frac{\partial}{\partial r}+\frac{\Omega_1}{r^2}\frac{\partial}{\partial \vartheta}+\frac{1}{r}\frac{\partial \psi_1}{\partial r}\frac{\partial}{\partial z}\right]\left[-\frac{1}{r}\frac{\partial \psi_2}{\partial z}\,\mathbf{i}_r+\frac{\Omega_2}{r}\,\mathbf{i}_\vartheta+\frac{1}{r}\frac{\partial \psi_2}{\partial r}\,\mathbf{i}_z\right]=0\end{aligned}\right. \tag{43}$$

so that the condition of the theorem discussed in section 2 is satisfied and (13) and (14) are also satisfied.

Now the conditions of integrability for the two motions given by (24) reduce to

$$\frac{\partial}{\partial t}\,(\text{curl }\mathbf{q}_i)=0 \tag{44}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\lambda_i}{r}\frac{\partial \psi_i}{\partial z}\right)=0 \tag{45}$$

and

(46) $$\frac{\partial}{\partial t}\left(\lambda_i \frac{\Omega_i}{r}\right) = 0.$$

From (41), (42) and (45)

(47) $$\frac{\partial}{\partial t}\left(\frac{1}{r}\,\frac{\partial \Omega_i}{\partial z}\right) = 0.$$

Integrating (47), we get

(48) $$\Omega_i = F_i(z) + \Phi_i(t) \qquad (i = 1,\ 2)$$

where $F_i(z)$ and $\Phi_i(t)$ are arbitrary differentiahle functions of z and t respectively. From (46), (48)

(49) $$\lambda_i = \frac{p_i(z)}{F_i(z) + \Phi_i(t)} \qquad (i = 1,\ 2)$$

where $p_i(z)$ is again an arbitrary differentiahle function of z. This equation gives the most general form which λ_1, λ_2 can have.

From (41), (42) (48) and (49)

(50) $$\frac{\partial \psi_2}{\partial z} = \frac{F_2(z) + \Phi_2(t)}{p_2(z)}\, F_1'(z)\,; \qquad \frac{\partial \psi_1}{\partial z} = \frac{F_1(z) + \Phi_1(t)}{p_1(z)}\, F_2'(z),$$

so that the general forms of velocity vectors are

(51) $$\mathbf{q}_1 = -\frac{1}{r}\left[\frac{F_1(z) + \Phi_1(t)}{p_1(z)}\right] F_2'(z)\, \mathbf{i}_r + \frac{F_1(z) + \Phi_1(t)}{r}\, \mathbf{i}_\vartheta,$$

(52) $$\mathbf{q}_2 = -\frac{1}{r}\left[\frac{F_2(z) + \Phi_2(t)}{p_2(z)}\right] F_1'(z)\, \mathbf{i}_r + \frac{F_2(z) + \Phi_2(t)}{r}\, \mathbf{i}_\vartheta.$$

Equations (48), (51) and (52) give the general solution of our problem. The functions F_i, Φ_i and p_i are however to be so chosen that ψ_i, Ω_i satisfy the first of the equations (41) and (42). From the first of these we get

(53) $$F_1' F_2' + F_1 F_2'' - F_1 F_2' \frac{p_1'}{p_1} + p_2\, p_1 = 0,$$

(54) $$\frac{F_2''}{F_2'} = \frac{p_1'}{p_1}\,.$$

On eliminating $\dfrac{p_1'}{p_1}$ from these equations

(55) $$F_1' F_2' + p_2\, p_1 = 0.$$

Integrating (54), we get

(56) $$\frac{F_2'}{p_1} = C.$$

Similarly from (42)

$$F_2' F_1' + p_1 p_2 = 0, \tag{57}$$

$$\frac{F_1'}{p_2} = D,$$

where C and D are arbitrary constants.

From (55), (56) and (57)

$$\frac{F_1'}{p_2} = D = \frac{1}{C}. \tag{58}$$

It is obvious that these results are consistent. Therefore equations (51) and (52) reduce to

$$\mathbf{q}_1 = -\frac{C}{r}\Big[F_1(z) + \Phi_1(t)\Big]\,\mathbf{i}_r + \frac{1}{r}\Big[F_1(z) + \Phi_1(t)\Big]\,\mathbf{i}_\vartheta, \tag{59}$$

$$\mathbf{q}_2 = -\frac{1}{Cr}\Big[F_2(z) + \Phi_2(t)\Big]\,\mathbf{i}_r + \frac{1}{r}\Big[F_2(z) + \Phi_2(t)\Big]\,\mathbf{i}_\vartheta, \tag{60}$$

so axially symmetric inviscid superposable flows of the given type are possible and are given by equations (59) and (60) when λ_1 and λ_2 are functions of z and t alone.

5. Steady axially symmetric viscous flows of the type curl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, curl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$ when λ_1, λ_2 are functions of r only.

The conditions of integrability from equations (13) and (14) of (5) are

$$r\frac{\partial \Omega}{\partial t} = -\frac{\partial(\Omega, \psi)}{\partial(z, r)} + \nu r D^2 \Omega, \tag{61}$$

$$\frac{1}{r}\frac{\partial}{\partial t} D^2 \psi = -\frac{\partial(r^{-2} D^2 \psi, \psi)}{\partial(z, r)} - \frac{2\Omega}{r^3}\frac{\partial \Omega}{\partial z} + \frac{\nu}{r} D^4 \psi. \tag{62}$$

As the flows are given to be steady and also self-superposable, since the conditions of theorem in section 2 are satisfied, the conditions of integrability reduce to

$$D^2 \Omega = 0, \tag{63}$$

$$D^4 \psi = 0. \tag{64}$$

In the present case, from equations (18) and (20) Ω_1, Ω_2, $\partial\psi_1/\partial r$ and $\partial\psi_2/\partial r$ are functions of r only. The conditions of integrability give

$$D^2 \Omega_i \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\Omega_i = 0 \qquad (i = (1,\ 2), \tag{65}$$

(66) $$D^4\psi_i \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \psi_i}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\psi_i\right) = 0 \qquad (i = 1,\ 2)$$

or

$$r^4 \psi_{rrrr} - 2r^3 \psi_{rrr} + 3r^2 \psi_{rr} - 3r \psi_r = 0.$$

The solutions of these equations are

(67) $$\Omega_1 = A_1 r^2 + B_1\,, \qquad \Omega_2 = A_2 r^2 + B_2,$$

(68) $$\psi_1 = C_1 r^2 + D_1 + E_1 r^4 + F_1 r^2 \log r\,, \qquad \psi_2 = C_2 r^2 + D_2 + E_2 r^4 + F_2 r^2 \log r.$$

From (21), (22) we get

(69) $$\lambda_2 = -\frac{D^2\psi_1}{\Omega_2} = \frac{\dfrac{\partial \Omega_1}{\partial r}}{\dfrac{\partial \psi_2}{\partial r}}\,; \qquad \lambda_1 = -\frac{D^2\psi_2}{\Omega_1} = \frac{\dfrac{\partial \Omega_2}{\partial r}}{\dfrac{\partial \psi_1}{\partial r}}\,.$$

Substituting from (67) and (68) for ψ_1, ψ_2, Ω_1 and Ω_2, we get

(70) $$(4E_1 r^2 + F_1)\left[(2C_2 + F_2) + 4E_2 r^2 + 2F_2 \log r\right] + A_1 (A_2 r^2 + B_2) = 0,$$

(71) $$(4E_2 r^2 + F_2)\left[(2C_1 + F_1) + 4E_1 r^2 + 2F_1 \log r\right] + A_2 (A_1 r^2 + B_1) = 0.$$

For (70) and (71) to hold identically for all values of r, we require

(72) $$\begin{cases} 8E_1C_2 + A_1A_2 = 0, & 8E_2C_1 + A_1A_2 = 0, \\ 2F_1C_2 + A_1B_2 = 0, & 2F_2C_1 + A_2B_1 = 0, \\ E_1E_2 = 0,\ F_1F_2 = 0, & E_1F_2 = 0,\ E_2F_1 = 0, \end{cases}$$

of which the solutions are

(73) $$\begin{cases} (i) & E_1 = 0, & F_1 = 0, & A_1 = 0, & F_2 = 0, & 2F_2C_1 + A_2B_1 = 0, \\ (ii) & F_1 = 0, & F_1 = 0, & A_1 = 0, & C_1 = 0, & B_1 = 0, \\ (iii) & E_1 = 0, & F_1 = 0, & A_1 = 0, & C_1 = 0, & A_2 = 0, \\ (iv) & E_1 = 0, & F_1 = 0, & A_2 = 0, & C_1 = 0, & B_2 = 0, \\ (v) & F_1 = 0, & F_1 = 0, & A_2 = 0, & E_2 = 0, & F_2 = 0, & A_1 = 0, \\ (vi) & E_1 = 0, & F_1 = 0, & A_2 = 0, & E_2 = 0, & F_2 = 0, & B_2 = 0, \\ (vii) & E_2 = 0, & F_2 = 0, & A_2 = 0, & C_2 = 0, & A_1 = 0, \\ (viii) & E_2 = 0, & F_2 = 0, & B_1 = 0, & C_2 = 0, & A_1 = 0, \\ (ix) & E_2 = 0, & F_2 = 0, & B_1 = 0, & A_2 = 0, & C_2 = 0, & B_2 = 0. \end{cases}$$

For finding λ_1, λ_2, we use

$$\lambda_1 = -\frac{8E_2r^2 + 2F_2}{A_1r^2 + B_1} = \frac{2A_2r}{2C_1r + 4E_1r^3 + 2F_1r \log r + F_1r}, \tag{74}$$

$$\lambda_2 = -\frac{8E_1r^2 + 2F_1}{A_2r^2 + B_2} = \frac{2A_1r}{2C_2r + 4E_2r^3 + 2F_2r \log r + F_2r}. \tag{75}$$

to get

$$\left\{\begin{array}{lll}
(i) & \lambda_1 = -\dfrac{2F_2}{B_1} = \dfrac{A_2}{C_1}, & \lambda_2 = 0, \\
(ii) & \lambda_1 = \infty, & \lambda_2 = 0 \\
(iii) & \lambda_1 = -\dfrac{8E_2r^2 + 2F_2}{B_1}, & \lambda_2 = 0, \\
(iv) & \lambda_1 = -\dfrac{8E_2r^2 + 2F_2}{A_1r^2 + B_1}, & \lambda_2 = \dfrac{2A_1}{2C_2 + 4E_2r^2 + 2F_2 \log r + F_2}, \\
(v) & \lambda_1 = 0, & \lambda_2 = 0, \\
(vi) & \lambda_1 = 0, & \lambda_2 = \dfrac{A_1}{C_2}, \\
(vii) & \lambda_1 = 0, & \lambda_2 = -\dfrac{8E_1r^2 + 2F_1}{B_2}, \\
(viii) & \lambda_1 = \dfrac{2A_2}{2C_1 + 4E_1r^2 + 2F_1 \log r + F_1}, & \lambda_2 = -\dfrac{8E_1r^2 + 2F_1}{A_2r^2 + B_2}, \\
(ix) & \lambda_1 = 0, & \lambda_2 = \infty.
\end{array}\right. \tag{76}$$

Thus we find that either (*i*) one of the λ' s is zero and other is constant or a function of r alone *i.e.* one motion is irrotational and the other is of the type curl $\mathbf{q} = \lambda\,\mathbf{q}$ with λ a constant or function of r alone or (*ii*) both λ' s are zero *i.e.* both motions are irrotational or (*iii*) both λ' s are function of r alone *i.e.* both motions are rotational or (*iv*) one of the λ' s is zero and the other is infinite *i.e.* there is only one motion which may be rotational or irrotational and the other motion vanishes. This may be regarded as a degenerate case.

The respective values of stream functions ψ_1 and ψ_2 and corresponding values of Ω_1 and Ω_2 are as follows :

$$
(77)\quad \begin{cases}
(i) & \psi_1=C_1r^2+D_1, \quad \Omega_1=B_1, \quad \psi_2=C_2r^2+D_2+F_2r^2\log r, \quad \Omega_2=A_2r^2+B_2, \\
(ii) & \psi_1=D_1, \quad \Omega_1=0, \quad \psi_2=C_2r^2+D_2+E_2r^4+F_2r^2\log r, \quad \Omega_2=A_2r^2+B_2, \\
(iii) & \psi_1=D_1, \quad \Omega_1=B_1, \quad \psi_2=C_2r^2+D_2+E_2r^4+F_2r^2\log r, \quad \Omega_2=B_2, \\
(iv) & \psi_1=D_1, \quad \Omega_1=A_2r^2+B_1, \quad \psi_2=C_2r^2+D_2+E_2r^4+F_2r^2\log r, \quad \Omega_2=0, \\
(v) & \psi_1=C_1r^2+D_1, \quad \Omega_1=B_1, \quad \psi_2=C_2r^2+D_2, \quad \Omega_2=B_2, \\
(vi) & \psi_1=C_2r^2+D_1, \quad \Omega_1=A_1r^2+B_1, \quad \psi_1=C_2r^2+D_2, \quad \Omega_2=0, \\
(vii) & \psi_1=C_2r^2+D_2+E_1r^4+F_1r^2\log r, \quad \Omega_1=B_1, \quad \psi_2=D_2, \quad \Omega_2=B_2, \\
(viii) & \psi_1=C_1r^2+D_1+E_1r^4+F_1r^2\log r, \quad \Omega_1=0, \quad \psi_2=D_2, \quad \Omega_2=A_2r^2+B_2, \\
(ix) & \psi_1=C_1r^2+D_1+F_1r^4+F_1r^2\log r, \quad \Omega_1=A_1r^2, \quad \psi_2=D_2, \quad \Omega_2=0.
\end{cases}
$$

6. Steady axially symmetric viscous flows of the type curl $\mathbf{q}_1=\lambda_2\,\mathbf{q}_2$, curl $\mathbf{q}_2=\lambda_1\,\mathbf{q}_1$ when λ_1, λ_2 are functions of z only.

In this case from (39) and (40) ψ_1, ψ_2, Ω_1 and Ω_2 are functions of z alone. As the flows are given to he steady and also selfsuperposable, since the conditions of the theorem proved in section 2 are satisfied, the conditions of integrability (61) and (62) reduce to

$$(78)\qquad \frac{\partial^2 \Omega_i}{\partial z^2}=0, \qquad (i=1,2).$$

$$(79)\qquad 2\left(\frac{\partial \psi_i}{\partial z}\frac{\partial^2 \psi_i}{\partial z^2}+\Omega_i\frac{\partial \Omega_i}{\partial z}\right)=\nu r^2\,\frac{\partial^4 \psi_i}{\partial z^4} \qquad (i=1,2).$$

Since the left hand side of above equation is a function of z alone, therefore from (80) we have

$$(80)\qquad \frac{\partial^4 \Psi_i}{\partial z^4},$$

$$(81)\qquad \frac{\partial^4 \psi_i}{\partial z}\frac{\partial^2 \psi_i}{\partial z^2}+\Omega_i\frac{\partial \Omega_i}{\partial z}=0.$$

On integrating (80) and (81), we get

$$(82)\qquad \psi_1=C_1z^3+D_1z^2+E_1z+F_1, \qquad \psi_2=C_2z^3+D_2z^2+E_2z+F_2$$

and

$$(83)\qquad \left(\frac{\partial \psi_1}{\partial z}\right)^2+\Omega_1^2=\text{const.}, \qquad \left(\frac{\partial \psi_2}{\partial z}\right)^2+\Omega_2^2=\text{const.}$$

On integrating (78), we get

$$(84)\qquad \Omega_1=A_1z+B_1, \qquad \Omega_2=A_2z+B_2.$$

From (83) and (84)

$$(85) \qquad \begin{cases} 3C_1 + A_1^2 = 0, & D_1 + A_1 B_1 = 0, \\ 3C_2 + A_2^2 = 0, & D_2 + A_2 B_2 = 0. \end{cases}$$

From (41) and (42)

$$(86) \qquad \begin{cases} \lambda_1 = \dfrac{\dfrac{\partial \Omega_2}{\partial z}}{\dfrac{\partial \psi_1}{\partial z}} = - \dfrac{\dfrac{\partial^2 \psi_2}{\partial z^2}}{\Omega_1}, \\[2ex] \lambda_2 = \dfrac{\dfrac{\partial \Omega_1}{\partial z}}{\dfrac{\partial \psi_2}{\partial z}} = - \dfrac{\dfrac{\partial^2 \psi_1}{\partial z^2}}{\Omega_2}. \end{cases}$$

Substituting from (82) and (84) for ψ_1, ψ_2, Ω_1 and Ω_2, we get

$$(87) \qquad \begin{cases} \lambda_1 = \dfrac{A_2}{3C_1 z^2 + 2D_1 z + E_1} = - \dfrac{6C_2 z + 2D_2}{A_1 z + B_1}, \\[2ex] \lambda_2 = \dfrac{A_1}{3C_2 z^2 + 2D_2 z + E_2} = - \dfrac{6C_1 z + 2D_1}{A_2 z + B_2}. \end{cases}$$

For (87) to hold identically for all values of z, we require

$$(88) \qquad \begin{cases} C_1 C_2 = 0, \quad C_1 D_2 = 0, \quad C_2 D_1 = 0, \\ 4D_1 D_2 + A_2 A_1 = -6C_1 E_2 = -6C_2 E_1, \\ A_2 B_1 + 2D_2 E_1 = 0, \quad A_1 B_2 + 2D_1 E_2 = 0. \end{cases}$$

The solutions of equations (85) and (88) are

$$(89) \qquad \begin{cases} (i) & A_1 = 0, \quad A_2 = 0, \quad C_1 = 0, \quad C_2 = 0, \quad D_1 = 0, \quad D_2 = 0, \\ (ii) & A_1 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad D_1 = 0, \quad E_1 = 0, \quad 3C_2 + \dfrac{D_2^2}{B_2^2} = 0. \end{cases}$$

The corresponding values of λ_1 and λ_2 are

$$(90) \qquad \begin{cases} (i) & \lambda_1 = 0 \quad \lambda_2 = 0, \\ (ii) & \lambda_1 = \infty, \quad \lambda_2 = 0. \end{cases}$$

Thus we find that either (*i*) both λ' s are zero *i.e.* both motions are irrotational or (*ii*) one of the λ' s is zero and the other is infinite *i.e.* there is only one motion which may he rotational or irrotational and the other motion vanishes. This may be regarded as a degenerate case.

The respective values of the stream functions ψ_1 and the ψ_2 and corresponding values of Ω_1 and Ω_2 are

$$(91)\quad \begin{cases} (i) & \psi_1 = E_1 z + F_1, \quad \psi_2 = E_2 z + F_2, \quad \Omega_1 = B_1, \quad \Omega_2 = B_2, \\ (ii) & \psi_1 = F_1, \quad \psi_2 = -\dfrac{D_2^2}{3B_2^2} z^3 + D_2 z^2 + E_2 z + F_2, \quad \Omega_1 = 0, \quad \Omega_2 = A_2 z + B_2 . \end{cases}$$

7. Some special inviscid axially symmetric flows of the type curl $\mathbf{q}_1 = \lambda_2 \mathbf{q}_2$, curl $\mathbf{q}_2 = \lambda_1 \mathbf{q}_1$.

Eliminating ψ_1, ψ_2 from (21) and (22), we get

$$(92)\qquad \frac{\partial^2 \Omega_1}{\partial r^2} - \left(\frac{1}{\lambda_2}\frac{\partial \lambda_2}{\partial r} + \frac{1}{r}\right)\frac{\partial \Omega_1}{\partial r} + \lambda_1 \lambda_2 \Omega_1 = 0,$$

$$(93)\qquad \frac{\partial^2 \Omega_2}{\partial r^2} - \left(\frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial r} + \frac{1}{r}\right)\frac{\partial \Omega_2}{\partial r} + \lambda_1 \lambda_2 \Omega_2 = 0.$$

Let

$$(94)\qquad \frac{1}{\lambda_2}\frac{\partial \lambda_2}{\partial r} + \frac{1}{r} = \frac{K}{r} \qquad i.e. \qquad \lambda_2 = A_1 r^{K-1},$$

$$(95)\qquad \frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial r} + \frac{1}{r} = \frac{K}{r} \qquad i.e. \qquad \lambda_1 = A_2 r^{-K-1},$$

so that

$$(96)\qquad r^2 \frac{d^2 \Omega_1}{dr^2} - Kr \frac{d\Omega_1}{dr} + A_1 A_2 \Omega_1 = 0,$$

$$(97)\qquad r^2 \frac{d^2 \Omega_2}{dr^2} + Kr \frac{d\Omega_2}{dr} + A_1 A_2 \Omega_2 = 0.$$

Therefore

$$(98)\qquad \Omega_1 = r^{\frac{K+1}{2}} \left[c_1 e^{\frac{1}{2}\sqrt{(K+1)^2 - 4A_1A_2}} + D_1 e^{-\frac{1}{2}\sqrt{(K+1)^2 - 4A_1A_2}} \right],$$

$$(99)\qquad \Omega_2 = r^{\frac{-K+1}{2}} \left[c_2 e^{\frac{1}{2}\sqrt{(K-1)^2 - 4A_1A_2}} + D_2 e^{-\frac{1}{2}\sqrt{(K-1)^2 - 4A_1A_2}} \right].$$

ψ_1 and ψ_2 are easily found from (21) and (22).

As a second case, let

$$(100)\qquad \frac{1}{\lambda_2}\frac{\partial \lambda_2}{\partial r} + \frac{1}{r} = \frac{K}{r} \qquad i.e. \qquad \lambda_2 = A_1 r^{K-1},$$

$$(101)\qquad \frac{1}{\lambda_1}\frac{\partial \lambda_1}{\partial a} + \frac{1}{r} = \frac{K'}{r} \qquad i.e. \qquad \lambda_1 = A_2 r^{K'-1}.$$

Now the equations (92) and (93) reduce to

$$(102)\qquad \frac{d^2 \Omega_1}{dr^2} - \frac{K}{r}\frac{d\Omega_1}{dr} + A_1 A_2 r^{K+K'-2}\, \Omega_1 = 0,$$

(103) $$\frac{d^2\Omega_2}{dr^2} - \frac{K'}{r}\frac{d\Omega_2}{dr} + A_1 A_2\, r^{K+K'-2}\,\Omega_2 = 0.$$

Now $y = x^{n\gamma}\; J_n(Bx^\gamma)$ satisfies the equation

(104) $$\frac{d^2 y}{dx^2} - \frac{2n\gamma - 1}{x}\frac{dy}{dx} + (B^2\gamma^2 x^{2\gamma-2})\, y = 0.$$

Therefore the solutions of equations (102) and (103) are

(105) $$\Omega_1 = r^{\frac{K+1}{2}}\; J_{\frac{K+1}{K+K'}}\left(\frac{2\sqrt{A_1A_2}}{K+K'}\, r^{\frac{K+K'}{2}}\right),$$

(106) $$\Omega_2 = r^{\frac{K'+1}{2}}\; J_{\frac{K'+1}{K+K'}}\left(\frac{2\sqrt{A_1A_2}}{K+K'}\, r^{\frac{K+K'}{2}}\right).$$

If A_1A_2 is negative, the solutions can be expressed in terms of modified BESSEL functions. Accordingly ψ_1 and ψ_2 are easily found from (21) and (22).

Eliminating ψ_1, ψ_2 from (41) and (42), we get

(107) $$\frac{d^2\Omega_1}{dz^2} - \frac{1}{\lambda_2}\frac{d\lambda_2}{dz}\frac{d\Omega_1}{dz} + \lambda_1\lambda_2\,\Omega_1 = 0,$$

(108) $$\frac{d^2\Omega_2}{dz^2} - \frac{1}{\lambda_1}\frac{d\lambda_1}{dz}\frac{d\Omega_2}{dz} + \lambda_1\lambda_2\,\Omega_2 = 0.$$

Let

(109) $$\lambda_1 = B_1 z^{K_1}, \qquad \lambda_2 = B_2 z^{K_2}.$$

Then from (108) and (109)

(110) $$\frac{d^2\Omega_1}{dz^2} - \frac{K_2}{z}\frac{d\Omega_1}{dz} B_1B_2\, z^{K_1+K_2}\,\Omega_1 = 0,$$

(111) $$\frac{d^2\Omega_2}{dz^2} - \frac{K_1}{z}\frac{d\Omega_2}{dz} B_1B_2\, z^{K_1+K_2}\,\Omega_2 = 0.$$

The solutions of these equations are

(112) $$\Omega_1 = z^{\frac{K_2+1}{2}}\; J_{\frac{K_2+1}{K_1+K_2+2}}\left(\frac{2\sqrt{B_1B_2}}{K_1+K_2+2}\, z^{\frac{K_1+K_2+2}{2}}\right),$$

(113) $$\Omega_2 = z^{\frac{K_1+1}{2}}\; J_{\frac{K_1+1}{K_1+K_2+2}}\left(\frac{2\sqrt{B_1B_2}}{K_1+K_2+2}\, z^{\frac{K_1+K_2+2}{2}}\right).$$

If $K_1 + K_1 = -2$

$$\Omega_1 = z^{\frac{K_2+1}{2}} \left[E_1\, e^{\frac{1}{2}\sqrt{(K_2+1)^2-4B_1B_2}} + F_1\, e^{-\frac{1}{2}\sqrt{K_2+1)^2-4B_1B_2}} \right], \tag{114}$$

$$\Omega_2 = z^{\frac{K_1+1}{2}} \left[E_2\, e^{\frac{1}{2}\sqrt{(K_1+1)^2-4B_1B_2}} + F_2\, e^{-\frac{1}{2}\sqrt{(K_1+1)^2-4B_1B_2}} \right]. \tag{115}$$

The respective values of the stream functions ψ_1 and ψ_2 can be obtained from (41) and (42) accordingly.

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ÖZET

λ_1, r ve t nin, λ_2 de z ve t nin fonksiyonları olmak üzere

$$\text{rot}\ \mathbf{q}_1 = \lambda_2\, \mathbf{q}_2\,; \quad \text{rot}\ \mathbf{q}_2 = \lambda_1\, \mathbf{q}_1$$

şeklinde, hız alanları hem kutupsal hem toroidal bileşenleri haiz, bir eksene nazaran simetrik, lzuciyetsiz ve üst üste tatbik edilebilen akışlar incelenmiştir.

λ_1 in sadece r nin ve λ_2 nin de sadece z nin fonksiyonları olmaları halinde luzuciyetii ve zamana tâbi olmayan ayni tipten akışlar da tetkik edilmiştir.

Ayrıca luzuciyetsiz bazı özel akışlar da gözden geçirilmiştir.